Cocycle deformations of operator algebras and noncommutative geometry

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Introduction

The subject of this thesis concerns deformation quantization for operator algebras, with considerations of some aspects of K-theory and certain concepts from non-commutative geometry.

Quantization is a concept that has strong motivation in physics, where it can be loosely described as being a process of passing from a classical mechanical system to a quantum mechanical system. This implies passing from a commutative algebra - the so called algebra of classical observables - to a non-commutative algebra of quantum observables. *Deformation* of a mathematical object can be thought of as providing a collection of similar objects depending on a parameter, such that the original object is obtained for an initial value of the parameter. *Deformation quantization* can then be understood as a process of quantization which produces a collection of new objects, each being a deformation of the original object, dependent on a parameter, typically a formal parameter $\hbar$ (alluding to Plank's constant). In a traditional implementation of this concept, one typically begins with a commutative algebra and produces a new algebra of formal power series expansions with respect to the formal parameter $\hbar$.

We shall here deal with deformation quantization in the context of operator algebras and what falls under the description of *strict* deformation quantization. This process takes a given algebra (not necessarily commutative) and yields a new algebra, referred to as a deformation of the original algebra. More generally one obtains a collection of new algebras, varying over a parameter space. This type of deformation quantization is referred to as *strict*, as the resulting objects are bonafide of the same type as the original object (e.g. C*-algebras), and do not merely consist of formally described elements such as power series in a formal parameter.

There is an important additional ingredient which greatly facilitates the strict deformation quantization of an operator algebra, and that is a group action. There is a rich body of theory for deformation quantization of operator algebras equipped with an action of a group by automorphisms, and this is the underlying deformation quantization principle we apply and further explore in the present thesis.

As a very rudimentary example to illustrate which implications the presence of some sort of group structure could have; assume we have a G-graded algebra $A = \oplus_{g \in G} A_g$ where G is some discrete group, and we are given a 2-cocycle $\omega : G \times G \longrightarrow \mathbb{C}^*$. One can define a new, so-called *twisted* multiplication $\times_\omega$ on A by the formula $a_g \times_\omega a_h = \omega(g,h) a_g a_h$. This new algebra structure could ideally lead to a C*-algebra A_ω, referred to as a deformation of A. Indeed some well-known examples of C*-algebras such as the irrational rotation algebras and twisted group C*-algebras do arise in this fashion.

In [52] M.A. Rieffel developed a framework for deformation quantization for actions of $\mathbb{R}^d$. In this setup, one considers a C*-algebra A equipped with a strongly continuous action $\alpha : \mathbb{R}^d \longrightarrow Aut(A)$ of *-automorphisms, and a real skew-symmetric matrix $J \in M_d(\mathbb{R})$. For $a, b \in A^\infty$, elements which are smooth for the action α, Rieffel establishes a new, so-called *deformed* product $a \times_J b = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \alpha_{Ju}(a)\alpha_v(b)e^{2\pi i\langle u,v\rangle}\, du\, dv$. Subsequently the necessary algebra structure and properties are laid forth, such as involution, C*-norm, representation theory etc., resulting in a C*-algebra A_J called the (Rieffel-) deformation of A.

A different approach based on twisting by group cocycles is explored by Kasprzak in [31], which extends Rieffel's framework. There the foundation is the theory of C*-algebraic crossed products by abelian groups and Landstad duality. Given a C*-algebra A with a strongly continuous action $\alpha : G \longrightarrow Aut(A)$ of a locally compact abelian group G, one has the standard G-product $(A \rtimes_\alpha G, \lambda, \widehat{\alpha})$ where $A \rtimes_\alpha G$ denotes the full (or equivalently reduced, as the group is abelian hence amenable) C*-crossed product, λ is the left regular representation and $\widehat{\alpha} : \widehat{G} \longrightarrow Aut(A \rtimes_\alpha G)$ is the dual action. Landstad duality theorem then states that A can be recovered as the Landstad algebra of said G-product, which is to say briefly that A is the generalized fixed-point subalgebra $A = (A \rtimes_\alpha G)^{\widehat{\alpha}}$. The idea is then to produce a *twisted* fixed-point algebra as follows. Suppose we are given a 2-cocycle $\psi : \widehat{G} \times \widehat{G} \longrightarrow \mathbb{T}$. This gives a continuous bounded function $\psi_\chi \in C_b(\widehat{G})$ for each $\chi \in \widehat{G}$ by $\psi_\chi(\sigma) = \psi(\chi, \sigma)$ for $\sigma \in \widehat{G}$. Define unitaries $U_\chi = \lambda(\psi_\chi)$ and define a new twisted dual action $\widehat{\alpha}^\psi : \widehat{G} \longrightarrow Aut(A \rtimes_\alpha G)$ by $\widehat{\alpha}^\psi_\chi(b) = U^*_\chi \widehat{\alpha}_\chi(b) U_\chi$. This gives rise to a new G-product $(A \rtimes_\alpha G, \lambda, \widehat{\alpha}^\psi)$, the so called twisted G-product. One then defines the deformed algebra A^ψ to be the Landstad algebra of the twisted G-product, or briefly $A^\psi = (A \rtimes_\alpha G)^{\widehat{\alpha}^\psi}$. The setup of Rieffel fits into this picture as the special case $G = \mathbb{R}^n$ and 2-cocycle $\psi(u, v) = e^{2\pi i\langle u,Jv\rangle}$.

Proceeding along the lines of group cocycle-based twisting in order to achieve deformation, one is naturally led to consider *coactions* instead of actions, to obtain a more general deformation quantization procedure. Coactions seem to be better geared towards general groups and not just abelian groups. Both of the previously mentioned approaches revolve around an action by an abelian group, and such an action is equivalent to a coaction by the group C*-algebra of the dual group, which would indicate that those approaches could be re-cast into a more general framework for deformation using coactions. It is also interesting to note that the notion of coaction was devised (among other things) to overcome the lack of a dual action for an action by a non-abelian group, as such a group would lack a (Pontryagin) dual group to begin with.

Having noted that Landstad duality plays a role in the approach of Kasprzak, it would be interesting to search for a counterpart in the theory of coactions and try to mimick the same approach. Indeed such a duality result holds by work of Quigg and Vaes (see e.g. [57]), and this allows us to carry over the construction as follows. Let $\delta : A \longrightarrow M(A \otimes C^*_r(G))$ be a coaction of a locally compact group G on the separable C*-algebra A. Then the crossed product $A \rtimes_\delta \widehat{G} \subset M(A \otimes K)$ is equipped with the dual action $\widehat{\delta} : G \longrightarrow Aut(A \rtimes_\delta \widehat{G})$, $\widehat{\delta}_g = Ad(1 \otimes \rho_g)$ where ρ denotes the usual right translation action. Suppose we are given a continuous 2-cocycle $\omega : G \times G \longrightarrow \mathbb{T}$. Define the twisted dual action $\widehat{\delta}^\omega_g = Ad(1 \otimes \rho^\omega_g)$ where ρ^ω denotes the twisted right translation action. By the Landstad duality result of Quigg and Vaes, it follows that $\widehat{\delta}^\omega$ is the dual action on a crossed product $A_\omega \rtimes_{\delta^\omega} \widehat{G}$ for some C*-subalgebra $A_\omega \subset M(A \rtimes_\delta \widehat{G})$ and some coaction δ^ω of G. We take this subalgebra A_ω to be the deformation of A. There is however an issue here, namely the assumption of continuity of the 2-cocycle ω is much too restrictive, and it would be preferable to work with Borel 2-cocycles. In the case of the group $\mathbb{R}^n$, any Borel cocycle is cohomologous to a continuous cocycle, even to a skew-symmetric bi-character. The choice of considering Borel cocycles for general groups, however, presents another problem: given a Borel 2-cocycle ω it is not immediate that $\widehat{\delta}^{\omega}$ is a well defined action, as $\omega(\cdot, g)$ does not immediately give an element of the multiplier algebra $M(C_0(G)) = C_b(G)$, so the Landstad duality result of Quigg and Vaes is not directly applicable in this situation. However, we remedy this by observing that the Landstad algebra, which we want to take as the definition of the deformed algebra, in the case of a continuous 2-cocycle is in fact recovered by applying slice maps to the image of a twisted Takesaki-Takai duality isomorphism, and the latter mapping is well defined also in the case of a Borel 2-cocycle, thus proposing a sensible definition of the deformed algebra also in this situation. This is the key idea in the approach to cocycle deformation we present in chapter 2.

It is also important to mention here the well known constructions of twisted group C*-algebras and twisted C*-crossed products. Packer and Raeburn gave a construction of twisted C*-crossed products in [49], where they considered a twisted dynamical system (A, G, α, u) where $u : G \times G \longrightarrow UM(A)$ is a Borel map with certain properties in relation to the action $\alpha : G \longrightarrow Aut(A)$ (this setup and the twisted convolution algebra was introduced by Busby and Smith in [5]). A twisted crossed product $A \rtimes_{\alpha,u} G$ is then introduced and studied. Among the chief results is the fact that the stabilized twisted action on $A \otimes K(L^2(G))$ becomes equivalent to a genuine action β and thus gives an isomorphism after stabilization $A \rtimes_{\alpha,u} G \otimes K(L^2(G)) \cong (A \otimes K(L^2(G)) \rtimes_{\beta} G$. Our ($\mathbb{C}$-valued) cocycle deformation is related to this construction by considering the coaction δ on $A \rtimes_{\alpha} G$ dual to the given action α, whence $(A \rtimes G)_{\omega} \cong A \rtimes_{\alpha,\omega} G$.

Twisted group C*-algebras can be seen as a special case of Packer-Raeburn's twisted C*-crossed products, although also introduced earlier in different forms. More recently Echterhoff, Luck and Phillips use twisted group C*-algebras to answer K-theoretic questions by homotopy arguments in [20]. In fact that paper can be described as the inspiration or precursor to the paper [62] by Yamashita which in turn motivated our approach to the cocycle deformation presented here.

As a small extra, we consider the cocycle deformation in the setting of Cuntz-Pimsner algebras. Recently Kaliszewski, Quigg and Robertson have considered ([28], [29]) coactions of Cuntz-Pimsner algebras with relations to coactions on the underlying modules. This generalizes previous work by Hao and Ng ([25]) on crossed products of C*-correspondences and their resulting Cuntz-Pimsner algebras. Altogether this provides for a suitable setting for our deformation approach to be applied. We do this in a very simplistic way, namely by considering only theta deformation, i.e. an action by $\mathbb{T}^n$ or equivalently a coaction by $\mathbb{Z}^n$ (actually the slightly more general case of a coaction by a discrete group). In this case we get much simplification, and need not the full force of [28], [29] and [25]. However, it would be interesting to pursue these ideas further to a more general extent.

The second question addressed in this thesis is the invariance of K-groups or more generally the KK-groups under deformation quantization.

It has long been a well known fact that the K-groups remain invariant under Rieffel deformation, i.e. given a C*-algebra A and its Rieffel deformation A_J, one has $K_*(A) \cong K_*(A_J)$. This was established by Rieffel in [53] by showing that A_J is stably isomorphic to a crossed product $(A \otimes K \otimes C_0(ker\ J)) \rtimes_\beta \mathbb{R}^n$ and then applying the Connes-Thom isomorphism in K-theory followed by suspension (dimension shift) and stability of the K-functor, which leads to the claimed isomorphism. The setup of Kasprzak lends a different route to a K-theory isomorphism in the case of an action by $\mathbb{R}^n$. As a consequence of a fundamental result in the theory of Landstad duality, one has the isomorphism of crossed products $A \rtimes_\alpha \mathbb{R}^n \cong A^\psi \rtimes_{\alpha^\psi} \mathbb{R}^n$ where in the latter crossed product the action α^ψ is just the original action restricted to the deformed algebra. The isomorphism $K_*(A) \cong K_*(A^\psi)$ then again follows from the Connes-Thom isomorophism for the crossed products.

In our cocycle deformation approach for C*-algebras with coactions, we present a different approach to a K-theory isomorphism which builds on the techniques given by Echterhoff, Luck and Phillips in [20] and which were also applied in the setting of deformation by Yamashita in [62]. Assuming the group G satisfies the Baum-Connes conjecture with coefficients, we show that given homotopic cocycles ω_0 and ω_1 the deformed algebras A_{ω_0} and A_{ω_1} have isomorphic K-groups. This is achieved by first establishing that a deformed algebra A_ω is stably isomorphic to the twisted dual (action) crossed product $(A \rtimes_\delta \widehat{G}) \rtimes_{\hat{\delta},\omega} G$, and thus passing to the equivalent question: given homotopic cocycles, whether the twisted crossed products $B \rtimes_{\alpha,\omega_0} G$ and $B \rtimes_{\alpha,\omega_1} G$ have the same K-theory groups. This is then approached by essentially using the homotopy between the cocycles to construct a crossed product bundle whose fibers are the aformentioned twisted crossed products, and apply the Packer-Raeburn stabilization trick to "untwist" the crossed product bundle. Subsequently one observes that the crossed product bundle is in fact a trivial bundle when restricting to any compact subgroup, so the evaluation map gives a KK-equivalence. We then apply the Green-Julg theorem to achieve the KK-equivalence for the group.

We also develop a different approach to a K-theory isomorphism under Rieffel deformation, namely through the theory of continuous fields of C*-algebras. Such methods are also well established, where central notions are typically homotopy equivalence, evaluation morphisms on continuous bundles and so on. We deal with KK-groups and work with the continuous field $(A_{tJ})_{t\in[0,1]}$ having the (canonical) algebra of sections $\Gamma((A_{tJ})_{t\in[0,1]}) = (C([0,1]) \otimes A)_J$ which was already studied to some extent in the original monograph of Rieffel ([52]). The aim is to deduce that the evaluation morphism $\pi_{t_0} : \Gamma((A_{tJ})_{t\in[0,1]}) \longrightarrow A_{t_0 J}$ of the continuous field gives a KK-equivalence element, which is an attractive feature as it implies isomorphisms between KK-groups (and thus also K-groups) of any two deformed algebras $A_{t_0 J}$ and $A_{t_1 J}$ in the continuous field, including in particular the undeformed (original) algebra $A = A_{0J}$. Our approach is based on combining theory of $C_0(X)$-algebras, RKK-theory and some more recent material on RKK-fibrations. The idea is roughly as follows: instead of working directly with the algebra of sections $\Gamma((A_{tJ})_{t\in[0,1]})$ of the continuous field of the Rieffel deformation, we use the equivalent setup of Kasprzak to deform the canonical $C[0,1])$-algebra $B = C([0,1]) \otimes A$ into B^ψ which is a $C([0,1])$-algebra being $C([0,1])$-linearly *-isomorphic to $\Gamma((A_{tJ})_{t\in[0,1]})$, i.e. with naturally isomorphic fibers $(B^\psi)_t \cong A_{tJ}$. By combining the isomorphism of crossed products $B \rtimes \mathbb{R}^n \cong B^\psi \rtimes \mathbb{R}^n$ with the Connes-Thom-Kasparov theorem in RKK-theory, which is the bundle-theoretic counterpart of the Connes-Thom isomorphism in KK-theory, we deduce that B^ψ is a so-called RKK-fibration. It is then quite straightforward, by fundamentals of RKK-fibrations ([21], [22]) to deduce that the evaluation morphism is a KK-equivalence.

The third investigation in this thesis is related to index theory in the vein of noncommutative geometry. We relate some index theoretical aspects of the theta deformation of a manifold to their classical counterparts in the equivariant setting.

Theta deformation is a special case of Rieffel deformation, but was introduced differently by Connes, Dubois-Violette, Landi etc. (see e.g. [16], [17]) as an interesting example or case study which was to be subjected to the tools of noncommutative geometry. Given a C*-algebra A with a strongly continuous action of $\sigma : \mathbb{T}^n \longrightarrow Aut(A)$ and a skew-symmetric matrix $\theta \in M_n(\mathbb{R})$, the theta deformed algebra A_θ is defined by Connes, Dubois-Violette et. al. as the fixed point subalgebra $(A \otimes C(\mathbb{T}^n_\theta))^{\sigma\otimes\tau^{-1}}$ for the diagonal action $\sigma \otimes \tau^{-1}$, where $C(\mathbb{T}^n_\theta)$ is the algebra of the noncommutative torus and τ is a natural gauge action. In the equivalent picture of Rieffel, this corresponds to the situation where the action of $\mathbb{R}^n$ factors through the compact quotient $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Recall that Atiyah-Singer index theory concerns the computation of the (Fredholm) index of an elliptic operator on a manifold. On the analytical side, this index is shown to be expressible as a complex residue of a certain zeta function involving the trace of the elliptic operator, and so is computable as a complicated expression involving the coefficients of the elliptic operator and the metric. On the algebraic side, the elliptic operator determines a K-homology class while the vector bundle (on which the operator acts) determines a K-theory class. These two dual elements pair together and result in the index, which is the essential statement of the index theorem. These considerations are effectively synthesized into the general framework of noncommutative geometry. Working with a unital *-algebra A, the K-homology groups $K^i(A)$, $i = 0, 1$, are built from Fredholm modules $(\mathcal{H}, F)$, and come with a natural pairing with K-theory, $K_i(A) \times K^i(A) \longrightarrow \mathbb{Z}$, e.g. $\langle [p], [(\mathcal{H}, F)]\rangle$, the so-called index pairing. The Chern(-Connes) character maps allow the passage from K-theory/homology to cyclic homology/cohomology, by $Ch_* : K_*(A) \longrightarrow HC_*(A)$ and $Ch^* : K^*(A) \longrightarrow HP^*(A)$, and equivalently express the index pairings in the cyclic theory as

$$\langle [p], [(\mathcal{H}, F)]\rangle = \langle Ch_*([p]), Ch^*([(\mathcal{H}, F)])\rangle.$$

A more "geometrical" refinement of Fredholm modules is that of spectral triple $(\mathcal{A}, \mathcal{H}, D)$, which can be considered a nicer representative of a K-homology class $[(\mathcal{H}, F)]$. Coming back to theta deformation; it turns out this is an isospectral deformation, which means that given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ for the *-algebra $\mathcal{A}$, there is a rather natural candidate $(\mathcal{A}_\theta, \mathcal{H}, D_\theta)$ as a spectral triple for the deformed *-algebra $\mathcal{A}_\theta$. It is then a natural question to compare the index pairings for the original and deformed spectral triples under the K-theory isomorphism between the original and deformed algebra, and it has been established by several authors that the index pairing remains invariant under deformation. We show this by using techniques and properties inherent to our presentation of theta deformation.

The Connes-Moscovici local index formula establishes an expression of the Chern character cocycle in terms of the residue of certain zeta functions associated to the spectral triple. This can be appropriately called a far reaching generalization of the Atiyah-Singer index theorem. We shall however restrict our attention to the JLO-cocycle expression of the Chern character cocycle. Connes, Dubois-Violette, Landi et.al consider in particular theta deformation of a manifold. One starts with a compact Riemannian spin manifold M (with fixed spin structure) and the classical spectral triple $(C^\infty(M), L^2(M,S), D)$ with the Dirac operator and L^2-spinors. Assuming there is an action $\sigma : \mathbb{T}^n \longrightarrow Aut(C^\infty(M))$, and a skew-symmetric matrix θ, one gets the deformed algebra denoted $C^\infty(M_\theta)$, and spectral triple $(C^\infty(M_\theta), L^2(M,S), D)$. We shall express the Chern character of this deformed spectral triple in terms of the equivariant Chern character for the $\mathbb{T}^n$-equivariant spectral triple. We do this by applying results of Chern and Hu ([10]) on G-equivariant Fredholm modules and their Chern characters. This uses the JLO-version of the Chern character and classical heat kernel asymptotic techniques. Our point of entry is basically an expansion of elements in the deformed algebra with respect to spectral subspaces (a Fourier-type decomposition) followed by a componentwise application of the equivariant Chern character.

The following is a brief description of the layout and structure of this thesis:

- *Part I: Chapter 1. Preliminaries.*
  This part is a collection of background material and a minimal account of some of the main facts which are referred to later. The topics included range from crossed products by actions and coactions to KK-theory, index theory and bundle theory. All of these are fairly standard topics and are treated in a wealth of literature. We give only a recollection of important concepts and state some important results. The referenced literature should be consulted for details.

- *Part II: Chapter 2. Cocycle deformation.*
  Building on the approach to Rieffel deformation as introduced in [31] and on deformation using coactions of discrete groups as in [62], we establish a method of deforming a C*-algebra equipped with a coaction of locally compact group G and a $\mathbb{T}$-valued Borel 2-cocycle.

- *Part II: Chapter 3. Rieffel deformation and KK-fibrations.*
  Using bundle techniques and some recently introduced notions of fibrations in RKK-theory, the continuous field of the Rieffel deformation is studied and the evaluation map is shown to give a KK-equivalence.

- *Part II: Chapter 4. Index theory of theta deformations.*
  The theta deformation of a spectral triple is studied. Invariance of the index is explained.

- *Part II: Chapter 5. Local index formula for theta deformations of manifolds.*
  The Chern character for the spectral triple $(C^\infty(M_\theta), L^2(S), D)$ corresponding to a theta deformation of a manifold is expressed in terms of the equivariant Chern character.

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Amandip S. Sangha
Oslo, March 24, 2014

Part I

Background

Chapter 1

Preliminaries

1.1 Actions, coactions and crossed products

Standard references for this section are [19], [60] and [50].

We assume throughout G to be a locally compact group with a given left-invariant Haar measure μ (we shall omit the notation in integrals), and modular function δ_G so that $\int_G f(s)\,ds = \delta_G(t)\int_G f(st)\,ds$ for $f \in C_c(G)$. Let α be an action of G on a separable C*-algebra A, i.e. $\alpha : G \longrightarrow Aut(A)$ is a strongly continuous group homomorphism. This data is typically referred to as a *C*-dynamical system* and denoted (A, G, α). One also says more briefly that (A, G, α) is an *action*. We shall restrict our presentation to reduced crossed products as these are the only ones used in the subsequent chapters.

Group algebra and comultiplication

On the compactly supported continuous functions $C_c(G)$ the convolution product $*$ is

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)\,ds, \quad f, g \in C_c(G).$$

The involution is given by

$$f^*(t) = \delta_G(t^{-1})\overline{f(t^{-1})}.$$

The L^1-norm is

$$||f||_1 = \int_G |f(s)|\,ds.$$

$C_c(G)$ equipped with the convolution product, involution and L^1-norm becomes an *involutive normed $*$-algebra*.

Definition 1.1.1. The Banach *-algebra $L^1(G)$ is the completion of $C_c(G)$ in the L^1-norm.

Definition 1.1.2. The *full group C*-algebra $C^*(G)$* is the enveloping C*-algebra of $L^1(G)$, i.e. it is the completion of $C_c(G)$ with respect to the largest C*-norm

$$||f||_{C^*} = \sup\{\pi(f) : \pi \text{ is a non-degenerate *-representation of } C_c(G)\}, \quad f \in C_c(G).$$

Let $U : G \longrightarrow B(\mathcal{H})$ be a strongly continuous unitary representation. Then

$$\pi_U : C_c(G) \longrightarrow B(\mathcal{H}), \quad \pi_U(f) = \int_G f(s)U_s\, ds$$

is a bounded non-degenerate *-represenation of $C_c(G)$ on $\mathcal{H}$.

The L^2-norm on $C_c(G)$ is $||f||_2 = \left(\int_G |f(s)|^2\, ds\right)^{\frac{1}{2}}$. The left regular representation $\lambda : G \longrightarrow B(L^2(G))$ and right regular representation $\rho : G \longrightarrow B(L^2(G))$ of G are the unitary representations given by

$$(\lambda_g \xi)(s) = \xi(g^{-1}s), \quad (\rho_g \xi)(s) = \delta_G(g)^{-1}\xi(sg), \quad \text{for } g \in G, \xi \in L^2(G).$$

The completion of the vector space $C_c(G)$ in the L^2-norm is the Hilbert space $L^2(G)$. Using the left regular representation λ, we get $\pi_\lambda : C_c(G) \longrightarrow B(L^2(G))$, $(\pi_\lambda(f)\xi)(t) = \int_G f(s)(\lambda_s \xi)(t)\, ds$.

Definition 1.1.3. The *reduced group C*-algebra* $C_r^*(G)$ is the completion of $C_c(G)$ in the norm $||f|| = ||\pi_\lambda(f)||$.

The unitaries λ_g belong to $M(C_r^*(G))$ for each $g \in G$. We have the comultiplication $\widehat{\Delta}$ on $C_r^*(G)$,

$$\widehat{\Delta} : C_r^*(G) \longrightarrow M(C_r^*(G) \otimes C_r^*(G)),$$

whose extension to $M(C_r^*(G))$ satisfies $\widehat{\Delta}(\lambda_g) = \lambda_g \otimes \lambda_g$, for $g \in G$. And there is a comultiplication Δ on $C_0(G)$ given by

$$\Delta : C_0(G) \longrightarrow M(C_0(G) \otimes C_0(G)), \quad \Delta(f)(g,h) = f(gh).$$

The fundamental unitary $W \in M(C_0(G) \otimes C_r^*(G))$ is the operator defined by

$$(W\xi)(s,t) = \xi(s, s^{-1}t), \quad \text{for } \xi \in L^2(G \times G).$$

After identifying $M(C_0(G) \otimes C_r^*(G))$ with the algebra of bounded strictly continuous maps $G \longrightarrow M(C_r^*)$, we have $W(g) = \lambda_g$. We have

$$W^*(1 \otimes f)W = \Delta(f) \ \text{ for } f \in C_0(G),$$

$$W(\lambda_g \otimes 1)W^* = \hat{\Delta}(\lambda_g) \ \text{ and } \ W^*(\rho_g \otimes 1)W = \rho_g \otimes \lambda_g \text{ for } \ g \in G.$$

We will also use the unitary $V = (\rho \otimes \iota)(W_{21}) \in M(\rho(C_r^*(G)) \otimes C_0(G))$. We have

$$V(f \otimes 1)V^* = \Delta(f) \ \text{ for } f \in C_0(G),$$

$$V^*(1 \otimes \rho_g)V = \rho_g \otimes \rho_g \ \text{ and } \ V(1 \otimes \lambda_g)V^* = \rho_g \otimes \lambda_g \text{ for } \ g \in G.$$

Actions

Definition 1.1.4. A *covariant homomorphism* of (A, G, α) into the multiplier algebra $M(D)$ for some C*-algebra D is a pair (π, U) where $\pi : A \longrightarrow M(D)$ is a nondegenerate *-homomorphism and $U : G \longrightarrow UM(D)$ is a strictly continuous homomorphism such that

$$\pi \circ \alpha_s = Ad_{U_s} \circ \pi$$

for all $s \in G$.

A covariant representation of (A, G, α) on a Hilbert space $\mathcal{H}$ is by definition a covariant homomorphism into $M(K(\mathcal{H})) = B(\mathcal{H})$ in the sense of the above definition.

We consider the *-algebra structure on $C_c(G, A)$ where

$$(f * g)(s) = \int_G f(t)\alpha_t(g(t^{-1}s))\,dt \quad \text{and} \quad f^*(s) = \delta_G(s^{-1})\alpha_s(f(s^{-1}))^*,$$

for $f, g \in C_c(G, A)$.

Definition 1.1.5. If (π, U) is a covariant homomorphism of (A, G, α) into $M(D)$, the *integrated form* is the homomorphism $\pi \rtimes U : C_c(G, A) \longrightarrow M(D)$,

$$(\pi \rtimes U)(f) = \int_G \pi(f(s))U_s\,ds. \tag{1.1.1}$$

It is a standard fact that the integrated form is well-defined and continuous with respect to the inductive limit topology, and $||(\pi \rtimes U)(f)|| \leqslant \int_G ||f(s)||\,ds$.

To any representation π of A on a Hilbert space $\mathcal{H}$, we associate a covariant representation $(\tilde{\pi}, \tilde{\lambda})$ of (A, G, α) on the Hilbert space $L^2(G, \mathcal{H})$ by

$$(\tilde{\pi}(a)\xi)(t) = \pi(\alpha_{t^{-1}}(a))\xi(t) \text{ and } (\tilde{\lambda}_s\xi)(t) = \xi(s^{-1}t), \quad a \in A, s, t \in G, \xi \in L^2(G, \mathcal{H}).$$

The integrated form is then $\tilde{\pi} \rtimes \tilde{\lambda} : C_c(G, A) \longrightarrow B(L^2(G, \mathcal{H}))$ given by

$$((\tilde{\pi} \rtimes \tilde{\lambda})(f)\xi)(t) = \int_G \pi(\alpha_{t^{-1}}(f(s)))\xi(s^{-1}t)\,ds.$$

Definition 1.1.6. The reduced crossed product $A \rtimes_\alpha G$ is the norm closure of the image of $\tilde{\pi_u} \rtimes \tilde{\lambda}$ where π_u is the universal representation of A.

Remark 1.1.7. Reduced crossed products are typically denoted $A \rtimes_{\alpha,r} G$ in the literature, whereas full crossed products are denoted $A \rtimes_\alpha G$. But in this thesis we consider only reduced crossed products, and we shall always denote them $A \rtimes_\alpha G$ omitting the r.

It can be shown that $A \rtimes_\alpha G$ is given (up to *-isomorphism) by the closure of the image of $\tilde{\pi} \rtimes \tilde{\lambda}$ for any faithful representation π of A.

The *canonical covariant homomorphism* (i_A, i_G) of (A, G, α) into $M(A \rtimes_\alpha G)$ is given by

$$(i_A(a)f)(s) = af(s), \quad (i_G(t)f)(s) = \alpha_t(f(t^{-1}s)). \tag{1.1.2}$$

A 1-cocycle for an action α of G on A is a strictly continuous family $U = \{u_g\}_{g \in G}$ of unitaries in $M(A)$ such that $u_{gh} = u_g\alpha_g(u_h)$. Given such a cocycle, we can define a new action α_U of G on A by $\alpha_{U,g} = u_g\alpha_g(\cdot)u_g^*$. The actions α and α_U are called exterior equivalent. We have an isomorphism $A \rtimes_\alpha G \cong A \rtimes_{\alpha_U} G$ defined by

$$\alpha(b) \mapsto \alpha_U(b), \quad 1 \otimes \rho_g \mapsto \alpha_U(u_g^*)(1 \otimes \rho_g).$$

Coactions

We shall only consider reduced coactions, and moreover we shall assume nondegeneracy.

Let $\widehat{\Delta} : C_r^*(G) \longrightarrow M(C_r^*(G) \otimes C_r^*(G))$ denote the comultiplication $\lambda_s \mapsto \lambda_s \otimes \lambda_s$, and let $W \in UM(C_0(G) \otimes C_r^*(G))$ be the fundamental unitary.

Definition 1.1.8. A *reduced coaction* of a locally compact group G on a C*-algebra A is an injective, nondegenerate homomorphism $\delta : A \longrightarrow M(A \otimes C_r^*(G))$ satisfying

(i) $\delta(a)(1 \otimes \lambda(x)) \in A \otimes C_r^*(G)$ for all $x \in C^*(G)$,

(ii) $(\delta \otimes 1)\delta = (1 \otimes \widehat{\Delta})\delta$,

(iii) $\overline{\delta(A)(1 \otimes C_r^*(G))} = A \otimes C_r^*(G)$.

The triple (A, G, δ) is referred to as a (reduced) coaction.

Definition 1.1.9. A *covariant homomorphism* of the (reduced) coaction (A, G, δ) into $M(D)$ is a pair (π, μ) of nondegenerate homomorphisms $\pi : A \longrightarrow M(D)$ and $\mu : C_0(G) \longrightarrow M(D)$ such that

$$(\pi \otimes 1)\delta(a) = Ad_{\mu \otimes 1}(W)(\pi(a) \otimes 1).$$

As we will only consider *reduced* coactions, we may omit this explicit reference and speak only of coactions.

Let (π, μ) be a covariant homomorphism of the coaction (A, G, δ) into $M(D)$. Then $C^*(\pi, \mu) = \overline{\pi(A)\mu(C_0(G))}$ is a C*-algebra. If $\pi : A \longrightarrow M(D)$ is a nondegenerate homomorphism, then $Ind(\pi) = ((\pi \otimes \lambda)\delta, 1 \otimes m)$ is a covariant homomorphism of (A, G, δ) into $M(D \otimes K(L^2(G)))$. This is said to be the covariant homormorphism *induced* from π.

Definition 1.1.10. Let (A, G, δ) be a coaction and $(j_A, j_G) = ((1_A \otimes \lambda)\delta, 1 \otimes m) = Ind(1_A)$. Then $A \rtimes_\delta \widehat{G} = C^*(j_A, j_G) \subset M(A \otimes K(L^2(G)))$ is the *crossed product* of the coaction (A, G, δ).

If (π, μ) is a covariant homomorphism of the coaction (A, G, δ) into $M(D)$, then there exists a unique nondegenerate homomorphism $\pi \rtimes \mu : A \rtimes_\delta \widehat{G} \longrightarrow M(D)$ such that $(\pi \rtimes \mu)j_A = \pi$ and $(\pi \rtimes \mu)j_G = \mu$. This is the universal property of the triple $(A \rtimes_\delta \widehat{G}, j_A, j_G)$.

A 1-cocycle for a coaction δ of G on A is a unitary $U \in M(A \otimes C_r^*(G))$ such that $(\iota \otimes \hat{\Delta})(U) = (U \otimes 1)(\delta \otimes \iota)(U)$. Given such a cocycle, we can define a new coaction δ_U by $\delta_U(a) = U\delta(a)U^*$. The coactions δ and δ_U are called exterior equivalent. The inner automorphism $\mathrm{Ad}\, U$ of $M(A \otimes K)$ defines an isomorphism of $A \rtimes_\delta \widehat{G}$ onto $A \rtimes_{\delta_U} \widehat{G}$, see [36, Theorem 2.9].

Duality

Definition 1.1.11. Let (A, G, δ) be a coaction and $A \rtimes_\delta \widehat{G}$ its crossed product. Then $\widehat{\delta} : G \longrightarrow Aut(A \rtimes_\delta \widehat{G})$ is the *dual action* of G on $A \rtimes_\delta \widehat{G}$ defined as follows. Equivalently expressing $\hat{\delta}$ as a homomorphism $A \rtimes_\delta \hat{G} \to M((A \rtimes_\delta \hat{G}) \otimes C_0(G))$, we have by definition

$$\hat{\delta}(\delta(a)) = \delta(a) \otimes 1, \quad \hat{\delta}(1 \otimes f) = 1 \otimes \Delta(f), \quad a \in A, f \in C_0(G).$$

It follows that

$$\hat{\delta}(x) = V_{23}(x \otimes 1)V_{23}^* \quad \text{for} \quad x \in A \rtimes_\delta \hat{G} \subset M(A \otimes K),$$

where V is the unitary $V = (\rho \otimes \iota)(W_{21}) \in M(\rho(C_r^*(G)) \otimes C_0(G))$.

A similar definition can be given for an action (A, G, α), where one gets a dual coaction $A \rtimes_\alpha G \longrightarrow M(A \rtimes_\alpha G \otimes C^*(G))$, but we shall omit this notion in this generality as we do not require it, but will rather discuss the special case of duality for actions by abelian groups in the next subsection.

We will simply denote by K the algebra of compact operators $K = K(\mathcal{H})$ when the Hilbert space $\mathcal{H}$ is implicitly understood, for instance $\mathcal{H} = L^2(G)$. Given a coaction δ of G on A we can consider the coaction $a \otimes T \mapsto \delta(a)_{13}(1 \otimes T \otimes 1)$ of G on $A \otimes K$, then take the 1-cocycle $1 \otimes W^*$ for this coaction (the cocycle identity means that $(\iota \otimes \hat{\Delta})(W) = W_{13}W_{12}$) and get a new coaction on $A \otimes K$. In order to lighten the notation we will denote this new coaction by δ_{W^*}. Then we have the following well known duality result.

Theorem 1.1.12. *Let (A, G, δ) be a coaction and $A \rtimes_\delta \widehat{G}$ its crossed product. Then*

$$(A \rtimes_\delta \widehat{G} \rtimes_{\hat{\delta}} G, \hat{\hat{\delta}}) \cong (A \otimes K, \delta_{W^*}).$$

Explicitly, the isomorphism is given by

$$\hat{\delta}(\delta(a)) = \delta(a) \otimes 1 \mapsto \delta(a), \quad \hat{\delta}(1 \otimes f) = 1 \otimes \Delta(f) \mapsto 1 \otimes f, \quad 1 \otimes 1 \otimes \rho_g \mapsto 1 \otimes \rho_g.$$

If we identify $A \otimes K$ with $\delta(A) \otimes K \subset M(A \otimes K \otimes K)$, then this isomorphism is simply $\operatorname{Ad} W_{23}$.

We now discuss in more detail how to recover A from $A \rtimes_\delta \widehat{G}$ for a coaction δ. Consider the homomorphism

$$\eta \colon A \rtimes_\delta \widehat{G} \to M((A \rtimes_\delta \widehat{G}) \otimes K) \subset M(A \otimes K \otimes K)$$

defined by $\eta(x) = W_{23}\hat{\delta}(x)W_{23}^*$. In other words, η is the composition of $\hat{\delta} \colon A \rtimes_\delta \widehat{G} \to M(A \rtimes_\delta \widehat{G} \rtimes_{\hat{\delta}} G)$ with the Takesaki-Takai duality isomorphism $A \rtimes_\delta \widehat{G} \rtimes_{\hat{\delta}} G \cong \delta(A) \otimes K$. Explicitly,

$$\eta(\delta(a)) = (\delta \otimes \iota)\delta(a), \quad \eta(1 \otimes f) = 1 \otimes f \in M((A \rtimes_\delta \widehat{G}) \otimes K)$$

From this we see that $\delta(A) \subset M(A \rtimes_\delta \widehat{G})$ is the closed linear span of elements of the form $(\iota \otimes \varphi)\eta(x)$ with $x \in A \rtimes_\delta \widehat{G}$ and $\varphi \in K^*$.

More generally, assume we are given an action α of G on a C*-algebra B and a nondegenerate homomorphism $\pi \colon C_0(G) \to M(B)$ such that $\alpha(\pi(f)) = (\pi \otimes \iota)\Delta(f)$. Put $X = (\pi \otimes \iota)(W)$ and consider the homomorphism

$$\eta \colon B \to M(B \otimes K), \quad \eta(x) = X\alpha(x)X^*.$$

Then by a Landstad-type result of Quigg [51, Theorem 3.3] and, more generally, Vaes [57, Theorem 6.7], the closed linear span $A \subset M(B)$ of elements of the form $(\iota \otimes \varphi)\eta(x)$, with $x \in B$ and $\varphi \in K^*$, is a C*-algebra, the formula $\delta(a) = X(a \otimes 1)X^*$ defines a coaction of G on A, and η becomes an isomorphism $B \cong A \rtimes_\delta \widehat{G}$ that intertwines α with $\hat{\delta}$.

Duality for abelian groups

Recall that for an action (A, G, α) with G abelian there is a natural action, *the dual action*, $\widehat{\alpha}$ of the dual group $\widehat{G}$ on $A \rtimes_\alpha G$ given by

$$\widehat{\alpha}_\chi(f)(s) = \chi(s)f(s), \quad f \in C_c(G, A).$$

The well known Takesaki-Takai duality theorem states that the iterated crossed product $A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G}$ is isomorphic to $A \otimes K(L^2(G))$. Recall also the standard fact that given an abelian group G, there is a one-to-one correspondence between coactions of G and strongly continuous actions of the dual group $\widehat{G}$. This equivalence is implemented by the Fourier transform isomorphism $Ad_U : C^*(G) \longrightarrow C_0(\widehat{G})$, which is given by conjugation by the unitary operator $U : L^2(G) \longrightarrow L^2(\widehat{G})$, $(U\xi)(g) = \int_G \xi(g)\bar{\chi}(g)\,dg$. The canonical extension $Ad_U : M(C^*(G)) \longrightarrow M(C_0(\widehat{G}))$ maps $\lambda_x \mapsto ev_x \in C_b(\widehat{G}) \subseteq M(C_0(\widehat{G}))$. If we let Δ' denote the comultiplication

$$\Delta' : C_0(\widehat{G}) \longrightarrow C_b(\widehat{G} \times \widehat{G}) = M(C_0(\widehat{G})) \otimes M(C_0(\widehat{G})), \quad \Delta'(f)(\chi_1, \chi_2) = f(\chi_1\chi_2),$$

for $f \in C_0(\widehat{G})$, then we get

$$\begin{aligned}(Ad_U \otimes Ad_U)\widehat{\Delta}(\lambda_x)(\chi_1, \chi_2) &= (ev_x \otimes ev_x)(\chi_1, \chi_2)\\ &= \chi_1(x)\chi_2(x) = ev_x(\chi_1)ev_x(\chi_2) = ev_x(\chi_1\chi_2)\\ &= \Delta'(ev_x)(\chi_1, \chi_2) = \Delta'(Ad_U(\lambda_x))(\chi_1, \chi_2),\end{aligned}$$

which shows $(Ad_U \otimes Ad_U)\widehat{\Delta} = \Delta' Ad_U$, i.e. the identification of the comultiplications under the Fourier isomorphism (Pontryagin duality). If $\alpha : \widehat{G} \longrightarrow Aut(A)$ is an action, then we get an injective nondegenerate homomorphism $\delta^\alpha : A \longrightarrow C_b(\widehat{G}, A) \subseteq M(A \otimes C_0(\widehat{G})$ by

$$\delta^\alpha(a)(\chi) = \alpha_\chi(a), \quad a \in A, \chi \in \widehat{G}.$$

It is straightforward to verify that (A, G, δ^α) is a coaction. Conversely, if $\delta : A \longrightarrow C_b(\widehat{G}, A) \subseteq M(A \otimes C_0(\widehat{G}))$ is any injective nondegenerate homomorphism which satisfies the coaction identity, then we get an action $\alpha^\delta : \widehat{G} \longrightarrow Aut(A)$

$$\alpha^\delta_\chi(a) = \delta(a)(\chi), a \in A, \chi \in \widehat{G}.$$

We now briefly recollect some of the basic notions introduced by Landstad on duality theory for crossed products by abelian groups (cf. [50] §7.8).

Definition 1.1.13. Let G be a locally compact abelian group and $\widehat{G}$ its Pontryagin dual group. Let B be a C*-algebra with a strict-continuous unitary-valued homomorphism $\lambda : G \longrightarrow M(B)$, and let $\widehat{\rho}$ be a strongly continuous action $\widehat{\rho} : \widehat{G} \longrightarrow Aut(B)$ satisfying

$$\widehat{\rho}_\chi(\lambda_\gamma) = \chi(\gamma)\lambda_\gamma$$

for all $\chi \in \widehat{G}$ and $\gamma \in G$. The triple $(B, \lambda, \widehat{\rho})$ is called a *G-product*. One also simply refers to B as a G-product when the rest is implicitly understood.

Given a G-product $(B, \lambda, \widehat{\rho})$, one may extend the given unitary representation λ to the *-homomorphism $\lambda : C^*(G) \longrightarrow M(B)$. Using the Fourier transform to identify $C^*(G) \cong C_0(\widehat{G})$ we write $\lambda : C_0(\widehat{G}) \longrightarrow M(B)$. This map is injective and we often omit λ from the notation.

Definition 1.1.14. Let $(B, \lambda, \widehat{\rho})$ be a G-product and let $x \in M(B)$. The element x satisfies the *Landstad conditions* if:

(i) $\widehat{\rho}_\chi(x) = x$ for all $\chi \in \widehat{G}$,

(ii) the map $G \ni \gamma \mapsto \lambda_\gamma x \lambda_\gamma^* \in M(B)$ is norm continuous,

(iii) $fxg \in B$ for all $f, g \in C_0(\widehat{G})$.

The set of elements satisfying the Landstad conditions turns out to be a subalgebra in $M(B)$. We shall refer to this subalgebra as the *Landstad algebra* of the G-product.

The foremost example of a G-product is produced by the crossed product construction. Indeed, given an abelian C*-dynamical system (B, G, α), the triple $(B \rtimes_\alpha G, \lambda, \widehat{\alpha})$ is a G-product whose Landstad algebra is precisely B. The following result states that any G-product arises in this way.

Theorem 1.1.15. *[50, Theorem 7.8.8] A C*-algebra B is a G-product $(B, \lambda, \widehat{\rho})$ if and only if there exists a C*-dynamical system (C, G, β) for which $B \cong C \rtimes_\beta G$ and $\widehat{\rho} = \widehat{\beta}$. The C*-dynamical system is unique up to covariant isomorphism, the C*-algebra C is just the associated Landstad algebra and $\beta = Ad\,\lambda$.*

1.2 KK-theory

We will only briefly recall some of the basic notions and definitions in KK-theory, and then collect a few of the needed results. We consider throughout separable C*-algebras.

Definition 1.2.1. Let A and B be C*-algebras. A Kasparov A-B module is a triple $\mathcal{E} = (E, \phi, F)$ where $E = E^0 \oplus E^1$ is a $\mathbb{Z}_2$-graded countably generated Hilbert B-module, $\phi : A \longrightarrow \mathcal{L}_B(E)$ is a *-homomorphism of degree 0 and $F \in \mathcal{L}_B(E)$ is an element of degree 1 such that

(i) $[F, \phi(a)] \in K_B(E)\ , \quad a \in A$

(ii) $(F^2 - 1)\phi(a) \in K_B(E), \quad a \in A$

(iii) $(F^* - F)\phi(a) \in K_B(E), \quad a \in A$

The set of Kasparov A-B modules is denoted $\mathbb{E}(A, B)$. We call these *even* Kasparov modules and also write $\mathbb{E}^0(A, B) = \mathbb{E}(A, B)$. Odd Kasparov modules are defined similarly, except there is no grading on the Hilbert module. The set of odd Kasparov modules is denoted $\mathbb{E}^1(A, B)$.
The set $\mathbb{D}^i(A, B)$, $i = 0, 1$ for even or odd Kasparov modules respectively, consists of those triples (E, ϕ, F) for which $[F, \phi(a)] = (F^2 - 1)\phi(a) = (F - F^*)\phi(a) = 0$ for every $a \in A$. These are called *degenerate modules*.

Definition 1.2.2 (Unitary equivalence of Kasparov modules)**.** Two Kasparov modules (E_0, ϕ_0, F_0) and (E_1, ϕ_1, F_1) are *unitarily equivalent*, written $(E_0, \phi_0, F_0) \approx_u (E_1, \phi_1, F_1)$, if there exists a unitary $u \in \mathcal{L}(E_0, E_1)$ (of degree 0 in the even case) such that

$$u\phi_0 = \phi_1 u, \quad uF_0 = F_1 u.$$

If $\phi : A \longrightarrow B$ is a *-homomorphism of C*-algebras, then the triple $(B, \phi, 0)$ is a Kasparov $A - B$-module.

Given (E_1, ϕ_1, F_1), a Kasparov A_1-B-module, and (E_2, ϕ_2, F_2), a Kasparov A_2-B-module, then $(E_1 \oplus E_2, \phi_1 \oplus \phi_2, F_1 \oplus F_2)$ is a Kasparov $A_1 \oplus A_2$-B-module.

Let $(E, \phi, F) \in \mathbb{E}^i(A, B)$ be a Kasparov A-B-module and $\psi : C \longrightarrow A$ be a *-homomorphism. Then $(E, \phi \circ \psi, F) \in \mathbb{E}^i(C, B)$ is a Kasparov C-B-module denoted $\psi^*(E, \phi, F)$. For $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{E}^i(A, B)$ and $\psi : C \longrightarrow A$ be a *-homomorphism, $\psi^*(\mathcal{E}_1 \oplus \mathcal{E}_2) \approx_u \psi^*(\mathcal{E}_1) \oplus \psi^*(\mathcal{E}_2)$.

If $\psi : B \longrightarrow C$ a *-homomorphism then using the internal tensor product $E \otimes_\psi C$ of Hilbert C*-modules, $(E \otimes_\psi C, \phi \otimes_\psi 1, F \otimes_\psi 1) \in \mathbb{E}^i(A, C)$ is a Kasparov A-C-module denoted $\psi_*(E, \phi, F)$. We have $\psi_*(\mathcal{E}_1 \oplus \mathcal{E}_2) \approx_u \psi_*(\mathcal{E}_1) \oplus \psi_*(\mathcal{E}_2)$.

Let $(E, \phi, F) \in \mathbb{E}^i(A, B)$ and suppose $\psi : B \longrightarrow C$ is a surjective *-homomorphism. Let $N_\psi = \{x \in E : \psi(\langle x, x\rangle_B) = 0\}$, and $q : E \longrightarrow E/N_\psi$ the quotient map. The quotient is a right C-module by $q(x)\psi(b) = q(xb)$, which we equip with the C-valued inner product $\langle q(x), q(y)\rangle_C = \psi(\langle x, y\rangle_B$. By completing we get a Hilbert C-module denoted E_ψ. One has a map $\mathcal{L}_B(E) \longrightarrow \mathcal{L}_C(E_\psi)$, $T \mapsto T_\psi$, where $T_\psi(q(x)) = q(T(x))$. Define $\phi_\psi : A \longrightarrow \mathcal{L}_C(E_\psi)$ by $\phi_\psi(a) = \phi(a)_\psi$. Now $(E_\psi, \phi_\psi, F_\psi) \in \mathbb{E}^i(A, C)$ is a Kasparov A-C-module denoted $(E, \phi, F)_\psi$.

If $(E, \phi, F) \in \mathbb{E}^i(A, B)$ and C is a C*-algebra, then $(E \otimes C, \phi \otimes 1, F \otimes 1) \in \mathbb{E}^i(A \otimes C, B \otimes C)$ is a Kasparov $A \otimes C$-$B \otimes C$-module denoted $\tau_C(E, \phi, F)$.

Definition 1.2.3 (Homotopic Kasparov modules)**.** (E_0, ϕ_0, F_0) and (E_1, ϕ_1, F_1) are *homotopic*, written $(E_0, \phi_0, F_0) \sim_h (E_1, \phi_1, F_1)$, if there exist a Kasparov module $(E, \phi, F) \in \mathbb{E}^i(A, C([0,1]) \otimes B)$ such that

$$(E, \phi, F)_{ev_0} \approx_u (E_0, \phi_0, F_0)$$

and

$$(E, \phi, F)_{ev_1} \approx_u (E_1, \phi_1, F_1),$$

where the pushouts are with respect to the evaluation morphisms $ev_t : C([0,1]) \otimes B \longrightarrow B$, $ev_t(f \otimes b) = f(t)b$, at $t = 0$ and $t = 1$ respectively.

Let $\mathcal{E} \in \mathbb{E}^i(A, B)$ and $f : B \longrightarrow C$ and $g : C \longrightarrow D$ be *-homomorphisms of C*-algebras. Then

$$g_*(f_*(\mathcal{E})) \approx_u (g \circ f)_*(\mathcal{E}) \in \mathbb{E}^i(A, D). \tag{1.2.1}$$

If $f : B \longrightarrow C$ a surjective *-homomorphism of C*-algebras, then

$$\mathcal{E}_f \approx_u f_*(\mathcal{E}). \tag{1.2.2}$$

One has that $\sim_h$ is an equivalence relation on $\mathbb{E}^i(A, B)$, and it is used to define the KK-groups.

Definition 1.2.4 (KK-groups)**.** $KK^i(A, B) = \mathbb{E}^i(A, B)/\sim_h$, $i = 0, 1$.

One also writes $KK(A, B) = KK^0(A, B)$.

Theorem 1.2.5. *$KK^i(A, B)$ is an abelian group with addition $[\mathcal{E}_1] + [\mathcal{E}_2] = [\mathcal{E}_1 \oplus \mathcal{E}_2]$. The following properties hold:*

(i) For $\mathcal{E}_1, \mathcal{E}_2 \in KK^i(A, B)$, $f : D \longrightarrow A$, $g : B \longrightarrow C$, then $f^*(g_*(\mathcal{E}_1)) \approx_u g_*(f^*(\mathcal{E}_1))$. And $\mathcal{E}_1 \approx_u \mathcal{E}_2$ implies $f^*(\mathcal{E}_1) \approx_u f^*(\mathcal{E}_2)$, $g_*(\mathcal{E}_1) \approx_u g_*(\mathcal{E}_2)$ and $\tau_C(\mathcal{E}_1) \approx_u \tau_C(\mathcal{E}_2)$.

(ii) *A *-homomorphism $f : A \longrightarrow B$ induces a group homomorphism*

$$f^* : KK^i(B, C) \longrightarrow KK^i(A, C)$$

by $f^*[\mathcal{E}] = [f^*\mathcal{E}]$ for $\mathcal{E} \in KK^i(A, B)$.

(iii) *A *-homomorphism $f : B \longrightarrow C$ induces a group homomorphism*

$$f_* : KK^i(A, B) \longrightarrow KK^i(A, C)$$

by $f_*[\mathcal{E}] = [f_*\mathcal{E}]$ for $\mathcal{E} \in KK^i(A, B)$.

(iv) *External tensor product gives a group homomorphism $\tau_D : KK^i(A, B) \longrightarrow KK^i(A \otimes D, B \otimes D)$, $[\mathcal{E}] \mapsto [\tau_D \mathcal{E}]$ for $\mathcal{E} \in KK^i(A, B)$.*

(v) *If $g_0, g_1 : D \longrightarrow B$ are homotopic *-homomorphisms, then $g_{0*} = g_{1*} : KK^i(A, D) \longrightarrow KK^i(A, B)$.*

(vi) *If $f_0, f_1 : A \longrightarrow D$ are homotopic *-homomorphisms, then $f_0^* = f_1^* : KK^i(D, B) \longrightarrow KK^i(A, B)$.*

For a C*-algebra A, its *suspension* SA is defined as $SA = C_0(\mathbb{R}) \otimes A$.

Theorem 1.2.6. *$KK^1(A, B) \cong KK^0(A, SB) \cong KK^0(SA, B)$.*

The KK-groups have the following relation to the K-groups: $KK^0(\mathbb{C}, B) = K_0(B)$ and $KK^1(\mathbb{C}, B) = K_1(B)$. Indeed, if we let B have unit 1, then for any $(E, \phi, F) \in \mathbb{E}^0(\mathbb{C}, B)$ one can find an equivalent representative $(\widetilde{E}, \widetilde{\phi}, \widetilde{F})$ such that $1 - \widetilde{F}\widetilde{F}^*$ and $1 - \widetilde{F}^*\widetilde{F}$ are compact projections. Then $Ker(\widetilde{F})$ and $Ker(\widetilde{F}^*)$ are finitely generated projective B-modules, thus $Ker(\widetilde{F}) - Ker(\widetilde{F}^*) \in K_0(B)$. This association provides the isomorphism. The case for $KK^1(\mathbb{C}, B)$ can be handled by applying suspensions.

It is customary notation to write $K^0(B) = KK(B, \mathbb{C})$ and $K^1(B) = KK^1(B, \mathbb{C})$. These are in fact the *K-homology* groups, which are often introduced by different (yet related) means, see e.g. Definition 1.3.3.

KK-product

The Kasparov product (or intersection product) is a pairing

$$KK(A_1, B_1 \otimes D) \times KK(D \otimes A_2, B_2) \longrightarrow KK(A_1 \otimes A_2, B_1 \otimes B_2) \tag{1.2.3}$$

which has the more convenient expression in case of $B_1 = A_2 = \mathbb{C}$ (and relabeling $A_1 = A$, $B_2 = B$)

$$KK(A, D) \times KK(D, B) \longrightarrow KK(A, B \tag{1.2.4}$$

$$(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}.$$

Let $1_A = [(A, id, 0)] \in KK(A, A)$.

Theorem 1.2.7 (Properties of the product)**.**

(i) *$\mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z}$, for $\mathbf{x} \in KK(A, D_1)$, $\mathbf{y} \in KK(D_1, D_2)$, $\mathbf{z} \in KK(D_2, B)$*

(ii) $\mathbf{x_1} \cdot (\mathbf{x_2} + \mathbf{x_3}) = \mathbf{x_1} \cdot \mathbf{x_2} + \mathbf{x_1} \cdot \mathbf{x_3}$

(iii) $(\mathbf{x_1} + \mathbf{x_2}) \cdot \mathbf{x_3}) = \mathbf{x_1} \cdot \mathbf{x_3} + \mathbf{x_2} \cdot \mathbf{x_3}$

(iv) $f^*(\mathbf{x} \cdot \mathbf{y}) = f^*(\mathbf{x}) \cdot \mathbf{y}$, *for a *-homomorphism* $f : A_1 \longrightarrow A$, $\mathbf{x} \in KK(A, B)$, $\mathbf{y} \in KK(B, C)$

(v) $g_*(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot g_*(\mathbf{y})$, *for a *-homomorphism* $g : C \longrightarrow C_1$, $\mathbf{x} \in KK(A, B)$, $\mathbf{y} \in KK(B, C)$

(vi) $h_*(\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot h^*(\mathbf{y})$, *for a *-homomorphism* $h : B_1 \longrightarrow B_2$, $\mathbf{x} \in KK(A, B_1)$, $\mathbf{y} \in KK(B_2, C)$

(vii) $g_*(\mathbf{x}) = \mathbf{x} \cdot \mathbf{g}$, *for a *-homomorphism* $g : D \longrightarrow B$, $\mathbf{x} \in KK(A, D)$, *where* $\mathbf{g} = [(B, g, 0)] \in KK(D, B)$

(viii) $f^*(\mathbf{y}) = \mathbf{f} \cdot \mathbf{y}$, *for a *-homomorphism* $f : A \longrightarrow D$, $\mathbf{y} \in KK(D, B)$, *where* $\mathbf{f} = [D, f, 0] \in KK(A, D)$

(ix) $\tau_D(\mathbf{x} \cdot \mathbf{y}) = \tau_D(\mathbf{x}) \cdot \tau_D(\mathbf{y})$, $\mathbf{x} \in KK(A, C)$, $\mathbf{y} \in KK(C, B)$

(x) $1_A \cdot \mathbf{x} = \mathbf{x} \cdot 1_D = \mathbf{x}$, $\mathbf{x} \in KK(A, D)$

KK-equivalence

Definition 1.2.8 (KK-equivalence)**.** An element $\mathbf{x} \in KK(A, B)$ is a *KK-equivalence* if there exists an element $\mathbf{y} \in KK(B, A)$ such that their Kasparov products satisfy $\mathbf{xy} = \mathbf{1}_A$ and $\mathbf{yx} = \mathbf{1}_B$. The algebras A and B are called *KK-equivalent* if there exists a KK-equivalence element in $KK(A, B)$.

If $\mathbf{x} \in KK(A, B)$ is a KK-equivalence, then for any D we get natural isomorphisms

$$\mathbf{x}\cdot : KK(B, D) \longrightarrow KK(A, D), \quad \mathbf{w} \mapsto \mathbf{x} \cdot \mathbf{w},$$

$$\cdot\mathbf{x} : KK(D, A) \longrightarrow KK(D, B), \quad \mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{x},$$

where the inverse map would be given by multiplication by $\mathbf{y} \in KK(B, A)$, the inverse KK-equivalence element of $\mathbf{x}$.

Homotopy equivalent C*-algebras are clearly KK-equivalent: if $\phi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ are *-homomorphisms such that $\phi \circ \psi \sim_h id_B$ and $\psi \circ \phi \sim_h id_A$, then the induced ϕ_* and ψ_* (and likewise ϕ^* and ψ^*) are inverse group homomorphisms, and moreover ϕ and ψ are inverse KK-equivalence elements.

If $\mathbf{x} \in KK(A, B)$ is a KK-equivalence element, then $\tau_D(x) \in KK(A \otimes D, B \otimes D)$ is a KK-equivalence.

A C*-algebra B is called *KK-contractible* if $KK(B, B) = 0$. This also implies

$$KK(B, D) = 0 = KK(D, B)$$

for any other C*-algebra D. As an example, any contractible C*-algebra (in the homotopy sense) is KK-contractible. In particular $Cone(B) = C_0([0, 1)) \otimes B$ is KK-contractible for any C*-algebra B.

Connes-Thom isomorphism

Let A be a separable C*-algebra with a continuous action $\alpha : \mathbb{R} \longrightarrow Aut(A)$. Given $f \in C_b(\mathbb{R}, \mathbb{C})$, we may think of f as a multiplier $f \in M(C^*(\mathbb{R})$, which in turn provides a multiplier denoted $F_f \in M(A \rtimes_\alpha \mathbb{R})$ in the natural way. If $f \in C_b(\mathbb{R}, \mathbb{C})$ is such that $\lim_{t\to+\infty} f(t) = 1$ and $\lim_{t\to-\infty} f(t) = 1$ then we shall call the corresponding multiplier $F_f \in M(A \rtimes_\alpha \mathbb{R})$ a *Thom operator* (cf. [3], §19.3).

Lemma 1.2.9. *Let F_f be a Thom operator. The element $[(A \rtimes_\alpha \mathbb{R}, i_\mathbb{R}, F_f)] \in KK^1(A, A \rtimes_\alpha \mathbb{R})$ is independent of the particular f, and is called the Thom element of $(A, \mathbb{R}, \alpha)$, denoted $\boldsymbol{t}_\alpha$.*

Lemma 1.2.10. *(i) If $g : (A, \mathbb{R}, \alpha) \longrightarrow (B, \mathbb{R}, \beta)$ is an equivariant homomorphism inducing the homomorphism $h : A \rtimes_\alpha \mathbb{R} \longrightarrow B \rtimes_\beta \mathbb{R}$ then $h_*(\boldsymbol{t}_\alpha) = g^*(\boldsymbol{t}_\beta)$,*

(ii) If $\gamma = \alpha \otimes 1$ is the diagonal action on $A \otimes B$, then $\boldsymbol{t}_\gamma = \boldsymbol{t}_\alpha \otimes \boldsymbol{1}_B = \tau_B(\boldsymbol{t}_\alpha)$,

(iii) If $A \rtimes_\alpha \mathbb{R} \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}}$ is identified with $A \otimes K$ by duality, then $\boldsymbol{t}_{\hat{\hat{\alpha}}} = \boldsymbol{t}_\alpha \otimes \boldsymbol{1}_K$,

(iv) The KK-element $\boldsymbol{u}_\alpha = \boldsymbol{t}_\alpha \otimes \boldsymbol{t}_{\hat{\alpha}} \in KK(A, A \rtimes_\alpha \mathbb{R} \rtimes_{\hat{\alpha}} \mathbb{R}) \cong KK(A, A)$, obtained by external tensor product of KK-elements, is independent of α.

Theorem 1.2.11. *Let A be a separable C*-algebra with a continuous action $\alpha : \mathbb{R} \longrightarrow Aut(A)$. Then $A \rtimes_\alpha \mathbb{R}$ is KK-equivalent to SA by the Thom element $\boldsymbol{t}_\alpha$.*

Proof. Briefly; using the above lemma one shows that $\mathbf{t}_\alpha$ has right KK-inverse $\mathbf{t}_{\hat{\alpha}}$ while on the other hand one may identify $\mathbf{t}_\alpha$ with $\mathbf{t}_{\hat{\hat{\alpha}}}$ which has $\mathbf{t}_{\hat{\alpha}}$ as a left KK-inverse. This shows that $\mathbf{t}_\alpha$ is a KK-equivalence element. □

Pimsner-Voiculescu six-term exact sequence

Theorem 1.2.12. *Let $\alpha \in Aut(A)$. There are six-term exact sequences*

$$\begin{array}{ccccc} KK(D,A) & \xrightarrow{1-\alpha_*} & KK(D,A) & \longrightarrow & KK(D, A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ KK^1(D, A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & KK^1(D,A) & \underset{1-\alpha_*}{\longleftarrow} & KK^1(D,A) \end{array} \tag{1.2.5}$$

and

$$\begin{array}{ccccc} KK(A,D) & \xleftarrow{1-\alpha^*} & KK(A,D) & \longleftarrow & KK(A \rtimes_\alpha \mathbb{Z}, D) \\ \downarrow & & & & \uparrow \\ KK^1(A \rtimes_\alpha \mathbb{Z}, D) & \longrightarrow & KK^1(A,D) & \underset{1-\alpha^*}{\longrightarrow} & KK^1(A,D) \end{array} \tag{1.2.6}$$

1.3 Index theory

Common references for the material in this section include [13], [12], [59] and [6].

K-homology and index pairing

In this section we recall the standard notions of index theory in the setting of noncommutative geometry.

Recall that a bounded operator $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is called *Fredholm* if $range(F) \subseteq \mathcal{H}_2$ is closed, and $ker(F) \subset \mathcal{H}_1$ and $coker(F) \subset \mathcal{H}_2$ are finite dimensional subspaces. Then its *Fredholm index* is defined as

$$Index(F) = dim\, ker(F) - dim\, coker(F).$$

For a simple example, the unilateral shift $S : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ is a Fredholm operator. If we let $\{e_1, e_2, e_3, \ldots\}$ denote the standard basis of $l^2(\mathbb{N})$, then $S(e_j) = e_{j+1}$. It is seen that $range(S) = \overline{span}\{e_2, e_3, \ldots\}$, $ker(S) = \{0\}$ and $l^2(\mathbb{N})/range(S) = \overline{span}\{e_1\}$. And the Fredholm index is $Index(S) = 0 - 1 = -1$.

Below we list some of the important properties of Fredholm operators and the Fredholm index.

Proposition 1.3.1. *(i) (Atkinson's theorem) Let $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$. Then F is Fredholm if and only if there is an operator $S : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ such that $FS - 1_{\mathcal{H}_2}$ and $SF - 1_{\mathcal{H}_1}$ are compact operators. Then S is also Fredholm.*

(ii) Let $Fred(\mathcal{H})$ denote the set of Fredholm operators $\mathcal{H} \longrightarrow \mathcal{H}$ for a given Hilbert space $\mathcal{H}$. Then the Fredholm index is locally constant on $Fred(\mathcal{H})$ and induces a bijection $Index : \pi_0(Fred(\mathcal{H})) \longrightarrow \mathbb{Z}$.

(iii) The Fredholm index satisfies $Index(F^*) = -Index(F)$, and $Index(FS) = Index(F) + Index(S)$, for Fredholm operators F and S.

(iv) $Index(F + T) = Index(F)$, for F Fredholm and T compact.

In particular it follows that the index of a self-adjoint Fredholm operator $F = F^*$ must be zero, as $index(F^*) = -index(F)$ implies $index(F) = -index(F)$.

Definition 1.3.2. A *Fredholm module* $(\mathcal{H}, \pi, F)$ over a separable C*-algebra A consists of a Hilbert space $\mathcal{H}$, a *-representation $\pi : A \longrightarrow B(\mathcal{H})$ and an operator $F : \mathcal{H} \longrightarrow \mathcal{H}$ such that for every $a \in A$

$$(F^2 - 1)\pi(a),\ (F - F^*)\pi(a) \text{ and } [F, \pi(a)]$$

are compact operators. The Fredholm module $(\mathcal{H}, \pi, F)$ is *graded* if there is an operator $\gamma : \mathcal{H} \longrightarrow \mathcal{H}$ such that $\gamma^2 = 1$, $\gamma = \gamma^*$, $\gamma F = F\gamma$ and $\gamma\pi(a) = \pi(a)\gamma$ for every $a \in A$. We write $(\mathcal{H}, \pi, F, \gamma)$ for a graded Fredholm module.

We will usually assume the algebra A to be unital and $\pi(1_A) = id_{\mathcal{H}}$, which then implies that the conditions on the Fredholm module are requiring $F^2 - 1$ and $F - F^*$ to be compact operators. Moreover, an odd Fredholm module is given by a unital representation π on $\mathcal{H}$ and an operator $F = 2P - 1 + K$ where K is a compact operator and P is a projection commuting with $\pi(A)$ modulo the compacts. An even Fredholm module is given by

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad \pi = \begin{pmatrix} \pi_+ & 0 \\ 0 & \pi_- \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix}$$

where $F_- = F_+^* + K$ for a compact K. This implies in particular that $F_+ : \mathcal{H}_+ \longrightarrow \mathcal{H}_+$ is Fredholm.

Given a Fredholm module $(\mathcal{H}, \pi, F)$ over A and a unitary operator $U : \mathcal{H}' \longrightarrow \mathcal{H}$, then $(\mathcal{H}', U^*\pi U, U^*FU)$ is also a Fredholm module over A, with grading $U^*\gamma U$ if γ was a grading for $(\mathcal{H}, \pi, F)$.

Let $\{(\mathcal{H}, \pi, F_t)\}_{t\in[0,1]}$ be a collection of Fredholm modules over A such that the function $t \mapsto F_t$ is norm continuous. We say this constitutes an operator homotopy between $(\mathcal{H}, \pi, F_0)$ and $(\mathcal{H}, \pi, F_1)$.

Given two Fredholm modules $(\mathcal{H}_1, \pi_1, F_1)$ and $(\mathcal{H}_2, \pi_2, F_2)$ over A, then $(\mathcal{H}_1 \oplus \mathcal{H}_2, \pi_1 \oplus \pi_2, F_1 \oplus F_2)$ is also a Fredholm module over A. The Fredholm module $(\mathcal{H}, \pi, F)$ is called *degenerate* if $F = F^*$, $F^2 = 1$ and $[F, \pi(a)] = 0$ for all $a \in A$.

Definition 1.3.3. The K-homology group $K^p(A)$ (for $p = 0, 1$) is the abelian group with one generator $[x]$ for each unitary equivalence class of Fredholm modules (graded if $p = 0$, ungraded if $p = 1$) with the following relations

(i) $[x_0] = [x_1]$ in $K^p(A)$, if x_0 and x_1 are operator homotopic Fredholm modules,

(ii) $[x_0] + [x_1] = [x_0 \oplus x_1]$ in $K^p(A)$, if x_0 and x_1 are modules of same parity,

(iii) $[x] = 0$, if x is a degenerate Fredholm module.

If $(\mathcal{H}, \pi, F)$ is a Fredholm module over A, then $(\mathcal{H}^k, \pi \otimes 1_k, F \otimes 1_k)$ is a Fredholm module over $M_k(A)$. If $p \in M_k(A)$ is a projection, then $p(F \otimes 1_k)p$ is also a Fredholm operator - this follows by using pVp as a quasi-inverse, where V is a quasi-inverse of $F \otimes 1_k$, Proposition 1.3.1 (i). Similarly if $u \in M_k(A)$ is a unitary, then $P_k u P_k - (1 - P_k)$ is a Fredholm operator, where $P_k = \frac{1}{2}(1 + F \otimes 1_k)$. These considerations enable the following definition of the pairing between K-theory and K-homology.

Definition 1.3.4. The pairing

$$K_0(A) \times K^0(A) \longrightarrow \mathbb{Z}$$

is given by

$$\langle [p], [(\mathcal{H}, \pi, F, \gamma)] \rangle = Index(p(F^+ \otimes 1_k)p : p\mathcal{H}^k \longrightarrow p\mathcal{H}^k).$$

The pairing

$$K_1(A) \times K^1(A) \longrightarrow \mathbb{Z}$$

is given by

$$\langle [u], [(\mathcal{H}, \pi, F)] \rangle = Index(P_k u P_k - (1 - P_k) : P_k\mathcal{H}^k \longrightarrow P_k\mathcal{H}^k)$$

with $P_k = \frac{1}{2}(1 + F) \otimes 1_k$.

In terms of KK-theory these index pairings $K_0(A) \times K^0(A) \longrightarrow \mathbb{Z}$ and $K_1(A) \times K^1(A) \longrightarrow \mathbb{Z}$ correspond to the respective KK-products $KK(\mathbb{C}, A) \times KK(A, \mathbb{C}) \longrightarrow KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$, $([p], [(\mathcal{H}, \pi, F, \gamma)]) \mapsto [p] \cdot [(\mathcal{H}, \pi, F, \gamma)]$ and
$KK^1(\mathbb{C}, A) \times KK^1(A, \mathbb{C}) \longrightarrow KK^2(\mathbb{C}, \mathbb{C}) \cong KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$, $([u], (\mathcal{H}, F)) \mapsto [u] \cdot [(\mathcal{H}, F)]$.

For any $p \geqslant 1$ the p-th Schatten ideal of the Hilbert space $\mathcal{H}$ is

$$\mathcal{L}^p(\mathcal{H}) = \{T \in B(\mathcal{H}) : Trace(|T|^p) < \infty\}.$$

Definition 1.3.5. A (normalized) Fredholm module $(\mathcal{H}, \pi, F)$ over a unital A is called $(p+1)$-summable if there is a dense subalgebra $\mathcal{A} \subseteq A$ such that $[F, a] \in \mathcal{L}^{p+1}(\mathcal{H})$ for every $a \in \mathcal{A}$.

Definition 1.3.6. Let $(\mathcal{H}, \pi, F)$ be a $(p+1)$-summable normalized Fredholm module over A. For each $n \geqslant p$ (of same parity as the Fredholm module) define cyclic cocycles

$$Ch_n(\mathcal{H}, F)(a_0, a_1, \ldots, a_n) = \frac{\lambda_n}{2} Trace(\gamma[F, a_0][F, a_1] \cdots [F, a_n]), \tag{1.3.1}$$

where $\gamma = 1$ in case of an odd Fredholm module and

$$\lambda_n = \begin{cases} (-1)^{n(n-1)/2}\Gamma(\frac{n}{2}+1) & \text{, even Fredholm module} \\ (-1)^{n(n-1)/2}\Gamma(\frac{n}{2}+1) & \text{, odd Fredholm module} \end{cases}$$

The Chern character $Ch(\mathcal{H}, F)$ is the class of these cocycles in periodic cyclic cohomology.

Theorem 1.3.7. *Let $(\mathcal{H}, \pi, F)$ be a finitely summable normalized Fredholm module over A. Then*

$$\langle [e], [(\mathcal{H}, \pi, F)] \rangle = \frac{1}{(n/2)!} Ch_n(\mathcal{H}, \pi, F)(e, e, \ldots, e)$$

for $[e] \in K_0(A)$ and n large enough and even, and

$$\langle [u], [(\mathcal{H}, \pi, F)] \rangle = \frac{1}{2^n \Gamma(\frac{n}{2}+1)} Ch_n(\mathcal{H}, \pi, F)(u^*, u, \ldots, u)$$

for $[u] \in K_1(A)$ and n large enough and odd.

Spectral triples

Definition 1.3.8. A *spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a *-algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$ as bounded operators, $\pi : \mathcal{A} \longrightarrow B(\mathcal{H})$, and a densely defined self-adjoint (unbounded) operator $\mathcal{D} : dom(\mathcal{D}) \longrightarrow \mathcal{H}$ such that

(i) $a \cdot dom(\mathcal{D}) \subset dom(\mathcal{D})$, $a \in \mathcal{A}$,

(ii) $[\mathcal{D}, a] = \mathcal{D}a - a\mathcal{D}$ is a bounded operator, $a \in \mathcal{A}$,

(iii) $a(1 + \mathcal{D}^2)^{-\frac{1}{2}}$ is a compact operator for every $a \in \mathcal{A}$.

Here elements $a \in \mathcal{A}$ are implicitly understood as operators $\pi(a) \in B(\mathcal{H})$. If there exists an operator $\gamma \in B(\mathcal{H})$ such that $\gamma = \gamma^*$, $\gamma^2 = 1$, $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$ and $\gamma a = a\gamma$ for every $a \in \mathcal{A}$, then the spectral triple is said to be even graded.

When we assume $\mathcal{A}$ to be unital and the unit to act as the identity on the Hilbert space, then $(1 + \mathcal{D}^2)^{-\frac{1}{2}}$ becomes a compact operator. To study (Fredholm) index properties related to $\mathcal{D}$, one does the following modifications. Define the Hilbert space $\mathcal{H}_1 = \{\xi \in \mathcal{H} : \mathcal{D}\xi \in \mathcal{H}\}$ with inner product $\langle \xi, \eta \rangle_1 = \langle \xi, \eta \rangle + \langle \mathcal{D}\xi, \mathcal{D}\eta \rangle$. It follows that $\mathcal{D} : \mathcal{H}_1 \longrightarrow \mathcal{H}$ is a bounded operator (with operator norm at most

1), and furthermore it is Fredholm, as $\mathcal{D}(1+\mathcal{D}^2)^{-1} : \mathcal{H} \longrightarrow \mathcal{H}_1$ acts as an inverse modulo compact operators. This follows from

$$\begin{aligned}\mathcal{D}\mathcal{D}(1+\mathcal{D}^2)^{-1} &= (1+\mathcal{D}^2-1)(1+\mathcal{D}^2)^{-1} = (1+\mathcal{D}^2)(1+\mathcal{D}^2)^{-1} - (1+\mathcal{D}^2)^{-1} \\ &= 1-(1+\mathcal{D}^2)^{-1},\end{aligned}$$

and $(1+\mathcal{D}^2)^{-1}$ is compact since $(1+\mathcal{D}^2)^{-1} = (1+\mathcal{D}^2)^{-\frac{1}{2}}(1+\mathcal{D}^2)^{-\frac{1}{2}}$ where $(1+\mathcal{D}^2)^{-\frac{1}{2}}$ was compact. The Fredholm index of $\mathcal{D}$ would immediately be 0 as $\mathcal{D}$ is self-adjoint, so one considers rather the "even" part of it. If $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$ is an even graded spectral triple, one uses the decomposition into even/odd parts $\mathcal{H}_1 = \mathcal{H}_1^+ \oplus \mathcal{H}_1^-$ etc, and defines $\mathcal{D}^+ = \frac{1-\gamma}{2}\mathcal{D}\frac{1+\gamma}{2} : \mathcal{H}_1^+ \longrightarrow \mathcal{H}^-$. This is just the decomposition

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{H}_1^+ \longrightarrow \mathcal{H}^-.$$

The operator $\mathcal{D}^+$ is also bounded and Fredholm, as $\mathcal{D}^+\mathcal{D}^{-1}(1+\mathcal{D}^2)^{-1}$ is the inverse modulo compacts. When we speak of the Fredholm index of $\mathcal{D}$, we refer to the Fredholm index of $\mathcal{D}^+ : \mathcal{H}_1^+ \longrightarrow \mathcal{H}^-$. One defines for each $s \geqslant 0$ the Sobolev space $\mathcal{H}_s = \{\xi \in \mathcal{H} : (1+\mathcal{D}^2)^{\frac{s}{2}}\xi \in \mathcal{H}\}$ with inner product $\langle \xi, \eta, \rangle_s = \langle \xi, \eta \rangle + \langle (1+\mathcal{D}^2)^{\frac{s}{2}}\xi, (1+\mathcal{D}^2)^{\frac{s}{2}}\eta \rangle$. Then $\mathcal{H}_s$ is a Hilbert space and one puts $\mathcal{H}_\infty = \cap_{s\geqslant 0}\mathcal{H}_s$. If $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma)$ is an even graded spectral triple and we write $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$, we can define $\mathcal{D}_s^+$ to be the restriction $\mathcal{D}^+ : \mathcal{H}_s \longrightarrow \mathcal{H}_{s-1}$, with $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{H}_1 = dom(\mathcal{D})$. For all $s \geqslant 1$ one has

$$Index(\mathcal{D}_s^+) = Index(\mathcal{D}^+). \tag{1.3.2}$$

This follows by noticing that the kernel and cokernel of $\mathcal{D}^+$ consist of elements of $\mathcal{H}_\infty$, hence they are independent of s.

Starting with a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ we get a Fredholm module $(\mathcal{H}, F_\mathcal{D})$ over $\overline{\mathcal{A}}$ by setting $F_\mathcal{D} = \mathcal{D}(1+\mathcal{D}^2)^{-\frac{1}{2}}$. This is usually checked under various simplifying assumptions (e.g. assuming $[|\mathcal{D}|, a]$ to be bounded for each $a \in A$).

A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called QC^k for $k \geqslant 1$, if $a \in dom(\delta^k)$ for all $a \in \mathcal{A}$, where $\delta(T) = [|\mathcal{D}|, T]$. The spectral triple is QC^∞ if it is QC^k for all $k \geqslant 1$.

Definition 1.3.9. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is *finitely summable* if there is some $s_0 > 0$ such that

$$Trace((1+\mathcal{D}^2)^{-\frac{s_0}{2}}) < \infty.$$

This estimate then holds for all $s > s_0$ and

$$p = \inf\{s \in \mathbb{R}_+ : Trace((1+\mathcal{D}^2)^{-\frac{s}{2}}) < \infty\}$$

is called the *spectral dimension* of the spectral triple.

Proposition 1.3.10. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a finitely summable QC^1 spectral triple with spectral dimension p. Then $(\mathcal{H}, F_\mathcal{D})$ is a $\lceil p \rceil + 1$-summable Fredholm module over $\mathcal{A}$.*

For $T \in \mathcal{K}(\mathcal{H})$ let $\mu_n(T)$ denote the n-th singular value of T, i.e. the n-th eigenvalue of $\sqrt{T^*T}$ when the eigenvalues are listed in non-increasing order. Let $\sigma_N(T) = \sum_{k=1}^N \mu_k(T)$ denote the N-th partial sum. Let $\mathcal{L}^{(p,\infty)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sigma_N(T) = O(N^{1-1/p})\}$ and $\mathcal{L}^{(1,\infty)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sigma_N(T) = O(\log N)\}$.

Definition 1.3.11. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is (n, ∞)-summable if $(1 + \mathcal{D}^2)^{-\frac{n}{2}} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$.

Definition 1.3.12. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is *θ-summable* if $Trace(e^{-t\mathcal{D}^2}) < \infty$, for all $t > 0$.

The McKean-Singer formula expresses an analytic formula for the index, and moreover is also used in establishing the algebraic expression of the index in terms of a pairing between K-theory and K-homology (via Chern character).

Theorem 1.3.13 (McKean-Singer)**.** *Let $\mathcal{D}$ be an unbounded self-adjoint operator with compact resolvent, let γ be a unitary such that $\gamma\mathcal{D} = -\mathcal{D}\gamma$, and let $f \in C(\mathbb{R})$ be an even function with $f(0) \neq 0$ and such that $f(\mathcal{D})$ is trace-class. With $\mathcal{D}^+ = (1 - P)\mathcal{D}P$, $P = \frac{1+\gamma}{2}$, then $\mathcal{D}^+ : P\mathcal{H} \longrightarrow (1 - P)\mathcal{H}$ is Fredholm and*

$$Index(\mathcal{D}^+) = \frac{1}{f(0)} Trace(\gamma f(\mathcal{D})).$$

A typical function used in conjunction with the McKean-Singer formula is $f(x) = e^{-tx^2}$ which in other words brings the heat operator $e^{-t\mathcal{D}^2}$ into consideration, with $Index(\mathcal{D}^+) = Trace(\gamma e^{-t\mathcal{D}^2})$.

Cyclic homology and cyclic cohomology

We briefly recall some of the features of cyclic homology and cyclic cohomology that are relevant for index theory. The index pairing between K-theory and K-homology equates to the pairing between cyclic homology and cyclic cohomology via the Chern characters

$$\langle [x], [(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle = -\frac{1}{\sqrt{2\pi i}} \langle [Ch_*(x)], [Ch^*(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle \tag{1.3.3}$$

where $[x] \in K_*(\mathcal{A})$ and $[(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ is the K-homology class of the spectral triple.

We use a certain bicomplex to define cyclic homology and cohomology. Let $C_m = \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes m}$ and $C^m = Hom(C_m, \mathbb{C})$ (multilinear functionals on C_m) where $\bar{\mathcal{A}} = \mathcal{A}/\mathbb{C}1$ with 1 being the unit of $\mathcal{A}$. Define coboundary operators $b : C^m \longrightarrow C^{m+1}$ and $B : C^m \longrightarrow C^{m-1}$ as

$$\begin{aligned}(b\phi_m)(a_0, a_1, \ldots, a_m, a_{m+1}) = &\sum_{j=0}^{m} (-1)^j \phi_m(a_0, a_1, \ldots, a_j a_{j+1}, \ldots, a_m) \\ &+ (-1)^{m+1} \phi(a_{m+1}a_0, a_1, \ldots, a_m),\end{aligned}$$

$$(B\phi_m)(a_0, a_1, \ldots, a_{m-1}) = \sum_{j=0}^{m-1} (-1)^{(m-1)j} \phi_m(1, a_j, a_{j+1}, \ldots, a_{m-1}, a_0, \ldots, a_{j-1})$$

for $\phi_m \in C^m$. These operators satisfy $b^2 = 0$, $B^2 = 0$ and $bB + Bb = 0$. A normalized (b, B)-cochain ϕ is a finite collection $\phi = (\phi_m)_{m=1,\ldots,M}$ where each $\phi_m \in C^m$. It is called a normalized (b, B)-cocycle if for all m, $b\phi_m + B\phi_{m+2} = 0$, in short $(b + B)\phi = 0$. A cochain ϕ is called even/odd if $\phi_m = 0$ for m even/odd.

The boundary operators $b^T : C_{m+1} \longrightarrow C_m$ and $B^T : C_{m-1} \longrightarrow C_m$ are the transpose of the coboundary operators (b, B). Namely, a (b^T, B^T)-chain is a (possibly infinite) collection $c = (c_m)_{m=1,2,\ldots}$ where each $c_m \in C_m$, and the pairing between a (b, B)-cochain $\phi = (\phi_m)_{m=1}^{M}$ and a (b^T, B^T)-chain $c = (c_m)_{m \geqslant 1}$ is given by

$$\langle \phi, c \rangle = \sum_{m=1}^{M} \phi_m(c_m)$$

which satisfies

$$\langle (b + B)\phi, c \rangle = \langle \phi, (b^T + B^T)c \rangle.$$

A (b^T, B^T)-chain $c = (c_m)$ is called a (b^T, B^T)-cycle if $b^T c_{m+2} + B^T c_m = 0$ for all m, written $(b^T + B^T)c = 0$ in short. A chain c is called even/odd if $c_m = 0$ for m even/odd. The chain $c = (c_m)_{m\, odd}$ is an odd normalized (b^T, B^T)-boundary if there is some even chain $e = (e_m)_{m\, even}$ such that $c_m = b^T e_{m+1} + B^T e_{m-1}$ for all m.

We omit the superscript on b^T and B^T and instead write b and B for both the coboundary and the boundary operators, as it should be clear from the context which operators are being considered, and thus referring simply to (b, B)-chains and (b, B)-cochains; (b, B)-cycles and (b, B)-cocycles.

Let $u \in \mathcal{A}$ be a unitary. The Chern character $Ch_*(u) = (Ch_{2j+1}(u))$ is the (infinite) odd (b, B)-cycle given by

$$Ch_{2j+1}(u) = (-1)^j j!\, u^* \otimes u \otimes u^* \otimes \cdots \otimes u, \quad (2j + 2 \text{ entries}). \tag{1.3.4}$$

Let $p \in \mathcal{A}$ be a projection. The Chern character $Ch_*(p) = (Ch_{2m}(p))$ is the even (b, B)-cycle given by

$$Ch_{2m}(p) = (-1)^m \frac{(2m)!}{2(m!)} (2p - 1) \otimes p^{\otimes 2m},\ Ch_0(p) = p. \tag{1.3.5}$$

As in general we may consider our cochains to consist of infinite sequences, it is natural to impose some decay condition on these sequences. One gets various types of cyclic cohomology depending on the decay condition imposed on the cochains.

Definition 1.3.14. Cyclic cohomology $HC^*(\mathcal{A})$ is the cohomology of the total complex of the (b, B)-bicomplex with respect to finitely supported cochains.

The periodic cyclic cohomology $HP^*(\mathcal{A})$ is the cohomology of the double complex

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots \\
\cdots \arrow[r, "B"] & C^3 \arrow[u, "b"] \arrow[r, "B"] & C^2 \arrow[u, "b"] \arrow[r, "B"] & C^1 \arrow[u, "b"] \arrow[r, "B"] & C^0 \arrow[u, "b"] \\
\cdots \arrow[r, "B"] & C^2 \arrow[u, "b"] \arrow[r, "B"] & C^1 \arrow[u, "b"] \arrow[r, "B"] & C^0 \arrow[u, "b"] & \\
\cdots \arrow[r, "B"] & C^1 \arrow[u, "b"] \arrow[r, "B"] & C^0 \arrow[u, "b"] & & \\
\cdots \arrow[r, "B"] & C^0 \arrow[u, "b"] & & &
\end{tikzcd}$$

As the Chern characters $Ch_*(u) = (Ch_{2j+1}(u))$ and $Ch_*(p) = (Ch_{2m}(p))$ are infinite sequences, there must be put some growth constraint on the cochains these elements are paired with, concerning the index pairing (1.3.3). We shall favor *entire* cochains, as the JLO-cocycle (section 1.3) is entire. A cochain $\phi = (\phi_m)_m$ is entire if the sequences $\sum_{k=0}^{\infty} ||\phi_{2k}|| \frac{z^k}{k!}$ and $\sum_{k=0}^{\infty} ||\phi_{2k+1}|| \frac{z^k}{k!}$ have infinite radius of convergence, where $||\phi_m|| = \sup\{|\phi_m(a_0, \ldots, a_m)| : ||a_j|| \leqslant 1\}$. This is the entire cyclic cohomology.

When given a spectral triple $(A, \mathcal{H}, \mathcal{D})$ for which the corresponding Fredholm module $(\mathcal{H}, F_{\mathcal{D}})$ happens to be finitely summable (e.g. by Proposition 1.3.10), then the element $Ch^*(\mathcal{A}, \mathcal{H}, \mathcal{D})$ occurring on the right hand side in (1.3.3) is $Ch^*(\mathcal{H}, F_{\mathcal{D}})$ from Definition 1.3.1. We are more interested in the θ-summable case, and so we shall take the JLO-cocycle as the representative of the Chern character, which is defined in the following section.

JLO-cocycle

The JLO-cocycle is a representative for the Chern character of a θ-summable spectral triple. It's defined for an even spectral triple by a collection of cochains $\{JLO_{2k}\}_{k\geqslant 0}$ and for an odd spectral triple by $\{JLO_{2k+1}\}_{k\geqslant 0}$. These are defined by

$$\begin{aligned} &JLO_{2k}(a_0, a_1, \ldots, a_{2k}) \\ &= \int_{\Delta_{2k}} Trace(\gamma a_0 e^{-t_1\mathcal{D}^2}[\mathcal{D}, a_1] e^{-(t_2-t_1)\mathcal{D}^2} \cdots e^{-(t_{2k}-t_{2k-1})\mathcal{D}^2}[\mathcal{D}, a_{2k}] e^{-(1-t_{2k})\mathcal{D}^2}) \\ &dt_1 \cdots dt_{2k}, \end{aligned} \tag{1.3.6}$$

and

$$\begin{aligned} &JLO_{2k+1}(a_0, a_1, \ldots, a_{2k+1}) \\ &= \sqrt{2\pi i} \int_{\Delta_{2k+1}} Trace(a_0 e^{-t_1\mathcal{D}^2}[\mathcal{D}, a_1] e^{-(t_2-t_1)\mathcal{D}^2} \cdots e^{-(t_{2k+1}-t_{2k-1})\mathcal{D}^2}[\mathcal{D}, a_{2k}] e^{-(1-t_{2k+1})\mathcal{D}^2}) \\ &dt_1 dt_1 \cdots dt_{2k+1}, \end{aligned} \tag{1.3.7}$$

where $\Delta_n = \{(s_1, \ldots, s_n) : 0 \leqslant s_1 \leqslant \ldots \leqslant s_n \leqslant 1\}$ denotes the standard n-simplex.

For a θ-summable spectral triple $(A, \mathcal{H}, \mathcal{D})$ the index pairing (1.3.3) is expressible as

$$\langle [p], [(A, \mathcal{H}, \mathcal{D})] \rangle = \langle Ch(p), JLO(A, \mathcal{H}, \mathcal{D}) \rangle = \sum_{k=0}^{\infty} JLO_{2k}(Ch_{2k}(p))$$

and similarly in the odd case.

For the classical spectral triple $(C^\infty(M), L^2(M, S), \mathcal{D})$ of a compact Riemannian spin manifold M with Dirac operator $\mathcal{D}$ acting on L^2-spinors, it is well known ([4]) that the Chern character of the spectral triple is given by

$$Ch_k(f_0, \ldots, f_k) = vol(\Delta_k)(2\pi)^{-n}(\frac{2}{i})^{\frac{n}{2}} \int_M \widehat{A}\, f_0\, df_1 \wedge \cdots \wedge df_k,$$

for $f_0, \ldots, f_k \in C^\infty(M)$.

Residue cocycle

Although we make no use of the residue cocycle and the Connes-Moscovici index theorem itself, we include a brief discussion here. The proofs of the Connes-Moscovici index theorem ([15]) were subsequently refined in [26] and [7], [8]. We give a very brief recollection as in [6].

In this section we let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ finitely summable spectral triple. Define $\mathcal{B}(\mathcal{A}) \subset B(\mathcal{H})$ to be the algebra of polynomials generated by $\delta^n(a)$ and $\delta^n([\mathcal{D}, a])$ for $a \in \mathcal{A}$ and $n \geqslant 0$. The spectral triple has *discrete dimension spectrum* $Sd \subseteq \mathbb{C}$ if Sd is a discrete set such that for all $b \in \mathcal{B}(\mathcal{A})$ the zeta function $Trace(b(1+\mathcal{D}^2)^{-z})$ is defined and holomorphic for large $Re(z)$ and analytically continues to $\mathbb{C}\backslash Sd$. The dimension spectrum is *simple* if the zeta function has poles of order at most one for all $b \in \mathcal{B}(\mathcal{A})$. The dimension spectrum is *finite* if there is a $k \in \mathbb{N}$ such that the zeta function has poles of order at most k for all $b \in \mathcal{B}(\mathcal{A})$. The dimension spectrum is *infinite* if it is not finite.

We fix some convenient notation: for $k \in \mathbb{N}^n$ let $|k| = k_1 + \ldots + k_m$ and

$$\alpha(k) = \frac{1}{k_1!k_2!\cdots k_m!(k_1+1)(k_1+k_2+2)\cdots(|k|+m)}.$$

Define the numbers $\widetilde{\sigma}_{n,j}$ and $\sigma_{n,j}$ by the identities

$$\prod_{j=0}^{n-1}(z+j+\frac{1}{2}) = \sum_{j=0}^{n} z^j\,\widetilde{\sigma}_{n,j}, \quad \prod_{j=0}^{n-1}(z+j) = \sum_{j=0}^{n} \sigma_{n,j}.$$

For an operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ write $T^{(n)} = [\mathcal{D}^2, [\mathcal{D}^2, [\cdots, [\mathcal{D}^2, T]\cdots]]]$ for the n-th iterated commutator of T with $\mathcal{D}^2$. The real number

$$q = \inf\{k \in \mathbb{R} : Trace((1+\mathcal{D}^2)^{-\frac{k}{2}}) < \infty\}$$

is called the *spectral dimension* of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. The spectral triple is said to have *isolated spectral dimension* if for all elements b of the form

$$b = a_0[\mathcal{D}, a_1]^{(k_1)}\cdots[\mathcal{D}, a_m]^{(k_m)}(1+\mathcal{D}^2)^{-\frac{m}{2}-|k|}, \quad a_j \in \mathcal{A} \tag{1.3.8}$$

the zeta functions

$$\zeta_b(z - \frac{1-q}{2}) = Trace(b(1+\mathcal{D}^2)^{-z+\frac{1-q}{2}})$$

have analytic continuations to a deleted neighborhood of $z = \frac{1-q}{2}$. For a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with isolated spectral dimension we define functionals τ_j on elements b of the form (1.3.8),

$$\tau_j(b) = res_{z=\frac{1-q}{2}}\left[(z - \frac{1-q}{2})^j \cdot \zeta_b(z - \frac{1-q}{2})\right].$$

Let $L = \{x+iy : 0 < x < \frac{1}{2}, y \in \mathbb{R}\}$, and $R_s(\lambda) = (\lambda - (1+s^2+\mathcal{D}^2))^{-1}$. We define the function-valued (b, B)-cocycles ϕ_m^r as follows. For m odd and $r > 0$ we put

$$\begin{aligned}&\phi_m^r(a_0, a_1, \ldots, a_m)\\ =&\frac{-2}{\Gamma(\frac{m+1}{2})}\int_0^\infty s^m\, Trace\left(\frac{1}{2\pi i}\int_L \lambda^{-\frac{q}{2}-r} a_0 R_s(\lambda)[\mathcal{D}, a_1]R_s(\lambda)\cdots[\mathcal{D}, a_m]R_s(\lambda)\,d\lambda\right) ds.\end{aligned} \tag{1.3.9}$$

For m even and $r > \frac{1}{2}$ we put

$$\phi_m^r(a_0, a_1, \ldots, a_m)$$
$$= \frac{(m/2)!}{m!} \int_0^\infty 2^{m+1} s^m \, Trace\left(\gamma \frac{1}{2\pi i} \int_L \lambda^{-\frac{q}{2}-r} a_0 R_s(\lambda)[\mathcal{D}, a_1] R_s(\lambda) \cdots [\mathcal{D}, a_m] R_s(\lambda)\, d\lambda\right)$$
$$ds. \tag{1.3.10}$$

Theorem 1.3.15. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd finitely summable QC^∞ spectral triple with spectral dimension $q \geqslant 1$. Let $N = \lfloor \frac{q}{2} \rfloor$ and $u \in \mathcal{A}$ unitary.*

(i) $Index(PuP) = \frac{1}{\sqrt{2\pi i}} res_{z=\frac{1-q}{2}} \left[\sum_{odd\, m=1}^{2N-1} \phi_m^r(Ch_m(u))\right]$. *The sum has at most a simple pole at $r = \frac{1-q}{2}$. The complex function-valued cochain $(\phi_m^r)_{odd\, m=1}^{2N-1}$ is a (b, B)-cocycle modulo functions holomorphic in a half-plane containing $r = \frac{1-q}{2}$*

(ii)

$$Index(PuP) =$$
$$\frac{1}{\sqrt{2\pi i}} res_{z=\frac{1-q}{2}} \left[\sum_{odd\, m=1}^{2N-1} \sum_{|k|=0}^{2N-1-m} \sum_{j=0}^{|k|+\frac{m-1}{2}} (-1)^{|k|+m} \alpha(k) \Gamma(\frac{m+1}{2}) \tilde{\sigma}_{|k|+\frac{m-1}{2}, j}\right.$$
$$\left. \cdot (r - \frac{1-q}{2})^j Trace\left(u^*[\mathcal{D}, u]^{(k_1)} [\mathcal{D}, u^*]^{(k_2)} \cdots [\mathcal{D}, u]^{(k_m)} (1 + \mathcal{D}^2)^{-\frac{m}{2} - |k| - r + \frac{1-q}{2}}\right)\right]$$

(iii) If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has isolated spectral dimension, then

$$Index(PuP) = \frac{1}{\sqrt{2\pi i}} \sum_{odd\, m=0}^{2N-1} \phi_m(Ch_m(u))$$

where

$$\phi_m(a_0, \ldots, a_m) = res_{r=\frac{1-q}{2}} \phi_m^r(a_0, \ldots, a_m) = \sqrt{2\pi i} \sum_{|k|=0}^{2N-1-m} (-1)^{|k|} \alpha(k) \cdot$$
$$\cdot \sum_{j=0}^{|k|+\frac{m-1}{2}} \tilde{\sigma}_{|k|+\frac{m-1}{2}, j} \tau_j \left(a_0 [\mathcal{D}, a_1]^{(k_1)} \cdots [\mathcal{D}, a_m]^{(k_m)} (1 + \mathcal{D}^2)^{-|k| + \frac{m}{2}}\right)$$

and $(\phi_m)_{odd\, m=1}^{2N-1}$ is a (b, B)-cocycle.

Theorem 1.3.16. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an even finitely summable QC^∞ spectral triple with spectral dimension $q \geqslant 1$. Let $N = \lfloor \frac{q+1}{2} \rfloor$ and $p \in \mathcal{A}$ a self-adjoint projection.*

(i) $Index(p\mathcal{D}^+p) = res_{z=\frac{1-q}{2}} \left[\sum_{even\, m=0}^{2N} \phi_m^r(Ch_m(p))\right]$. *The sum has at most a simple pole at $r = \frac{1-q}{2}$. The complex function-valued cochain $(\phi_m^r)_{even\, m=0}^{2N}$ is a (b, B)-cocycle modulo functions holomorphic in a half-plane containing $r = \frac{1-q}{2}$*

(ii)

$$
\begin{aligned}
&Index(p\mathcal{D}^+p) = \\
&res_{r=\frac{1-q}{2}}\left[\sum_{even\, m=0}^{2N}\sum_{|k|=0}^{2N-m}\sum_{j=1}^{|k|+\frac{m}{2}}(-1)^{|k|+\frac{m}{2}}\alpha(k)\widetilde{\sigma}_{|k|+\frac{m}{2},j}\right.\\
&(r-\frac{1-q}{2})^j Trace\left(\gamma(2p-1)[\mathcal{D},p]^{(k_1)}[\mathcal{D},p]^{(k_2)}\cdots[\mathcal{D},p]^{(k_m)}\right.\\
&\left.\left.(1+\mathcal{D}^2)^{-\frac{m}{2}-|k|-r+\frac{1-q}{2}}\right)\right]
\end{aligned}
$$

(iii) If $(\mathcal{A},\mathcal{H},\mathcal{D})$ has isolated spectral dimension then

$$Index(p\mathcal{D}^+p) = \sum_{m=even\,0}^{2N}\phi_m(Ch_m(u))$$

where $\phi_0(a_0) = res_{r=\frac{1-q}{2}}\phi_0^r(a_0) = \tau_{-1}(\gamma a_0)$ *and for* $m \geqslant 2$

$$
\begin{aligned}
\phi_m(a_0,\ldots,a_m) &= res_{r=\frac{1-q}{2}}\,\phi_m^r(a_0,\ldots,a_m) = \sum_{|k|=0}^{2N-m}(-1)^{|k|}\alpha(k)\cdot \\
&\cdot\sum_{j=1}^{|k|+\frac{m}{2}}\sigma_{|k|+\frac{m}{2},j}\tau_{j-1}\left(\gamma a_0[\mathcal{D},a_1]^{(k_1)}\cdots[\mathcal{D},a_m]^{(k_m)}(1+\mathcal{D}^2)^{-|k|+\frac{m}{2}}\right)
\end{aligned}
$$

and $(\phi_m)_{even\, m=0}^{2N}$ *is a* (b,B)*-cocycle.*

Theorem 1.3.17. *For a QC^∞ finitely summable spectral triple with isolated spectral dimension, the functionals $(\phi_m)_{m=P,P+2,\ldots,2N-P}$, $P = 0,1$, is a (b,B)-cocycle that represents the Chern character. We call this (b,B)-cocycle the residue cocycle.*

Equivariant Chern character

This section contains some prerequisite material from [10] on the equivariant Chern character (as the JLO cocycle) in entire cyclic cohomology. Our terminology will differ slightly from that of [10], e.g. we will speak of the spectral triple directly rather than the Fredholm module.

Definition 1.3.18. Let G be a compact Lie group. A G-equivariant θ-summable spectral triple $(\mathcal{A},\mathcal{H},D)$ is a spectral triple (as in Definition 1.3.8) satisfying additionally

(i) $\mathcal{A}$ is a unital G-algebra, i.e. $\tau : G \longrightarrow Aut(\mathcal{A})$ is an action by continuous automorphisms,

(ii) $\mathcal{H}$ is a graded Hilbert space with grading operator γ, and $\rho : G \longrightarrow B(\mathcal{H})$ is a unitary group representation, with induced G-action on $B(\mathcal{H})$ given by $\rho_*(g)(P) = \rho(g)P\rho(g)^{-1}$,

(iii) $\mathcal{A}$ is represented on $\mathcal{H}$ as $\mu : A \longrightarrow B(\mathcal{H})$, and μ is even-graded and G-equivariant, i.e. $\mu(\tau(g)a) = \rho_*(g)(a)$,

(iv) D is an unbounded odd-graded self-adjoint G-invariant operator, $D\rho(g) = \rho(g)D$ for each $g \in G$,

(v) $Trace(e^{-D^2}) < \infty$.

The G-equivariant Θ-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is p-summable if the last condition of the definition is replaced by $(D+i)^{-1} \in \mathcal{L}^p(\mathcal{H})$.

We now recall the G-equivariant entire cyclic cohomology. Let

$$C^n_G = Hom_G(A^{\otimes(n+1)}, C(G)),$$

where $C^n_G = 0$ for $n < 0$. There is an induced G-action τ_* on C^n_G,

$$\tau_*(g)\phi(a^0, \ldots, a^n)(h) = \phi(\tau(g)^{-1}a^0, \ldots, \tau(g)^{-1}a^n)(g^{-1}hg).$$

The coboundary operators (b, B) are the same as in the previous section. The G-equivariant cyclic cohomology groups of $\mathcal{A}$ are then defined as

$$HC^n_G(\mathcal{A}) = H^n(C^{*,*}, b, B)$$

where $C^{n,m} = C^{n-m}$ is the (b, B)-bicomplex.

We have the even G-equivariant cochains $C^{even}_G = \{(\phi_{2n})_{n\in\mathbb{N}} : \phi_{2n} \in C^{2n}_G(\mathcal{A})\}$ and the odd G-equivariant cochains $C^{odd}_G = \{(\phi_{2n+1})_{n\in\mathbb{N}} : \phi_{2n+1} \in C^{2n+1}_G(\mathcal{A})\}$. An even cochain is called entire if the radius of convergence of $\sum_n \frac{||\phi_{2n}||z^n}{n!}$ is infinity. Likewise an odd cochain is entire if the radius of convergence of $\sum_n \frac{||\phi_{2n+1}||z^n}{n!}$. We denote the set of entire cochains $C_G(\mathcal{A})_{ent}$. The G-equivariant entire cyclic cohomology of $\mathcal{A}$ is the cohomology of the short complex

$$C^{even}_G(\mathcal{A})_{ent} \longrightarrow C^{odd}_G(\mathcal{A})_{ent} \longrightarrow C^{even}_G(\mathcal{A})_{ent}$$

Definition 1.3.19. The equivariant Chern character $ch^G_*(\mathcal{A}, \mathcal{H}, D) = \{ch^G_k(D)\}_{k\geqslant 0}$ is defined by

$$\begin{aligned} &ch^G_k(D)(a^0, a^1, \ldots, a^k)(g) \\ &= \int_{\Delta_k} Trace_s(a^0 e^{-s_1 D^2}[D, a^1]e^{-(s_2-s_1)D^2}[D, a^2] \cdots e^{-s_k - s_{k-1}}[D, a^k]e^{-(1-s_k)D^2} g)\, ds. \end{aligned} \tag{1.3.11}$$

Let M be an even-dimensional spin manifold and G a connected compact Lie group acting on M by Riemannian isometries. Let D denote the invariant Dirac operator, i.e. commuting with the isometries. Let $g \in G$ and denote by $M^g = \{x \in M : gx = x\}$ the corresponding fixed-point submanifold. We recall the two central quantities Ω and Θ from [10]. The matrix Ω of differential forms refers simply to the curvature matrix (for the Riemannian connection), which can be decomposed into a tangential and normal part, $\Omega^\top$ and $\Omega^\perp$, respectively, with respect to the submanifold M^g. The skew-symmetric real matrix Θ expresses locally the action by g (in terms of rotation angles). We refer to [10] for the details. The main results we will use from [10] are the following. One introduces a parameter t to replace D^2 with tD^2 and obtains an equivariant entire cyclic cocycle $ch^G_*(\sqrt{t}D) = \{ch^G_k(\sqrt{t}D)\}_{k\geqslant 0}$. This cocycle is cohomologous to ch^G_* in the equivariant entire cyclic cohomology (cf. [10, Remark 2.6 (iii)]). The main result is

Theorem 1.3.20. *[10] The equivariant Chern character of the invariant Dirac operator in the equivariant cyclic cohomology is given by*

$$\begin{aligned}\lim_{t\to 0^+} ch_k^G(\sqrt{t}D)(f_0,\dots,f_k)(g) = \frac{1}{k!(2\pi i)^{k/2}} \int_{M^g} & f_0 df_1 \wedge df_2 \wedge \cdots \wedge df_k \wedge \\ & \widehat{A}(TM^g) \wedge Pf(2\sinh(\frac{\Omega}{4\pi} + \frac{i\Theta}{2})(\nu(M^g)))^{-1},\end{aligned} \tag{1.3.12}$$

where k is even, $g \in G$, $f_i \in C^\infty(M)$ and M^g is the fixed-point set that is the disjoint union of a finite number of even-dimensional totally geodesic submanifolds of M.

1.4 Bundle theory

The material here can be found in [18], [33], [30], [21] and [22].

Continuous fields

Definition 1.4.1 (Continuous field of C*-algebras)**.** Let X be a compact Hausdorff space. A *continuous field of C*-algebras* (cf. [18, 10.3.1]) is a collection of C*-algebras $(D_x)_{x\in X}$ together with a distinguished set $\Gamma = \Gamma((D_x)_{x\in X}) \subseteq \prod_{x\in X} D_x$, the set of *sections*, satisfying

(i) Γ is a linear subspace

(ii) For each point $x \in X$ the set $\{y(x) : y \in \Gamma\}$ is dense in D_x

(iii) For each section $y \in \Gamma$ the function $x \mapsto ||y(x)||$ is continuous

(iv) For any element $y \in \prod_{x\in X} D_x$, if for every $x \in X$ and $\epsilon > 0$ there exists an element $y' \in \Gamma$ such that $||y(x) - y'(x)|| < \epsilon$ throughout some neighborhood of x, then $y \in \Gamma$.

The C*-algebra $\Gamma((D_x)_{x\in X})$, with pointwise product, involution and the norm $||y|| = \sup_{x\in X} ||y(x)||$, is *the algebra of continuous sections* of the continuous field $(D_x)_{x\in X}$, and each D_x is called a *fiber* of the continuous field.

Example 1.4.2 (Trivial field)**.** Let A be a C*-algebra and X a compact Hausdorff space. Consider $\Gamma = C(X) \otimes A \subseteq \prod_{x\in X} A$. This defines the *trivial continuous field* corresponding to A over the compact space X.

Bundles

The next notion removes the requirement of compactness of the base space and also makes it more viable to study various product structures in this setting (e.g. tensor products, crossed products by actions etc.). When the base space X is compact, a continuous field and a C*-bundle are essentially equivalent.

Definition 1.4.3. A *C*-bundle* ([33, 1.1]) over the locally compact Hausdorff space X is a triple $\mathcal{D} = (X, \pi_x : D \longrightarrow D_x, D)$ where D is a C*-algebra, the *bundle algebra*, and for each $x \in X$, D_x is a C*-algebra, the *fiber*, satisfying

(i) For each $x \in X$, $\pi_x : D \longrightarrow D_x$ is a *-epimorphism

(ii) For each $y \in D$, $||y|| = \sup_{x\in X} ||\pi_x(y)||$

(iii) For each $f \in C(X)$ and $y \in D$, there is an element $fy \in D$ for which $\pi_x(fy) = f(x)\pi_x(y)$

(iv) The bundle is *continuous* if in addition: for each $y \in D$ the function $x \mapsto ||\pi_x(y)||$ is an element of $C_0(X)$.

Example 1.4.4. The trivial C*-bundle is $(X, ev, C_0(X, A))$ with each fiber being $A_x = A$, and the morphism is just evaluation at $x \in X$, $ev_x(f) = f(x)$ for $f \in C_0(X, A)$.

Given a continuous C*-bundle $\mathcal{D} = (X, \pi, D)$ and another C*-algebra B, there are natural C*-bundles $\mathcal{D} \otimes B$ and $\mathcal{D} \otimes_{max} B$, with bundle algebras $D \otimes B$ and $D \otimes_{max} B$, respectively.

Suppose a continuous C*-bundle $\mathcal{D} = (X, \pi, D)$ is given and let G be a locally compact group. For each $x \in X$, let $\alpha^x : G \longrightarrow Aut(D_x)$, $g \mapsto \alpha^x_g$, be a continuous action by *-automorphisms. If for each $a \in D$ and $g \in G$ there is an element $\alpha_g(a) \in D$ such that $\pi_x(\alpha_g(a)) = \alpha^x_g(\pi_x(a))$, then $\{\alpha^x\}_{x\in X}$ is called a *continuous field of actions of* G and α is said to be a continuous action of G on the bundle algebra D, $g \mapsto \alpha_g$. We also say that G acts *fibrewise*. Consequently, for each $x \in X$, $\pi_x : D \longrightarrow D_x$ is a G-homomorphism and so we get natural a homomorphism $\pi^{G,r}_x : D \rtimes_\alpha G \longrightarrow D_x \rtimes_{\alpha^x, r} G$. Hence we get the crossed product C*-bundle $(X, \pi^{G,r}, D \rtimes_\alpha G)$. Continuity is not present in general, though the upper semi-continuity of these C*-bundles is dealt with similarly to the tensor product bundles.

Lemma 1.4.5. *([33, Remark 2.6.2]) Let $\mathcal{A} = (X, \pi_x : A \longrightarrow A_x)$ be a continuous C*-bundle and G be an amenable group acting fibrewise $\alpha^x : G \longrightarrow Aut(A_x)$ continuously. Then $A \rtimes_\alpha G$ is a continuous C*-bundle.*

$C_0(X)$-algebras

Definition 1.4.6. Let X be a locally compact Hausdorff space. A C*-algebra B is called a $C_0(X)$-algebra (cf. [30, 1.5]) if there is a non-degenerate *-homomorphism $\Phi_B : C_0(X) \longrightarrow ZM(B)$. For each $x \in X$, letting $I_x = \{f \in C_0(X) : f(x) = 0\}$ be the ideal of functions vanishing at x, then $I_xB \subseteq B$ is an ideal, and the quotient $B_x = B/(I_xB)$ is called the *fiber* over x. The quotient map $q_x : B \longrightarrow B_x$ is called the *evaluation map* at $x \in X$.

A *-homomorphism $\varphi : A \longrightarrow B$ between two $C_0(X)$-algebras is *fiber wise* if $\varphi \circ \Phi_A = \Phi_B \circ \varphi$. In this case φ induces *-homomorphism $\varphi_x : A_x \longrightarrow B_x$ for each $x \in X$.

An action by a locally compact group $\alpha : G \longrightarrow Aut(B)$ is a *fiber wise action* if $\alpha_g : B \longrightarrow B$ is fiber wise. The crossed product $B \rtimes_\alpha G$ is also a $C_0(X)$-algebra with $\Phi_{B\rtimes_\alpha G} = \iota \circ \Phi_B$ where $\iota : ZM(B) \longrightarrow ZM(B \rtimes_\alpha G)$ is the canonical embedding.

Let $f : Y \longrightarrow X$ be a continuous map between locally compact spaces. The pullback construction gives a $C_0(X)$-algebra structure on $C_0(Y)$, since $f^* : C_0(X) \longrightarrow C_b(Y)$ and $C_b(Y) = ZM(C_0(Y))$, we let $\Phi_{C_0(Y)} : C_0(X) \longrightarrow ZM(C_0(Y))$, $\Phi_{C_0(Y)}(k) = f^*(k)$ be the pointwise multiplication operator by the pullback

$$\Phi_{C_0(Y)}(k)h = f^*(k)h$$

for $k \in C_0(X)$, $h \in C_0(Y)$.

Given a $C_0(X)$-algebra B, a locally compact space Y and $f : Y \longrightarrow X$ a continuous map, the *pullback* $f^*(B)$ of B along f is the $C_0(Y)$-algebra

$$f^*(B) = C_0(Y) \otimes_{C_0(X)} B. \tag{1.4.1}$$

The balanced tensor product in (1.4.1) is by definition the quotient of $C_0(Y) \otimes B$ by the ideal generated by

$$\{\Phi_{C_0(Y)}(k)g \otimes b - g \otimes \Phi_B(k)b \mid \quad g \in C_0(Y),\, b \in B,\, k \in C_0(X)\}.$$

The $C_0(Y)$-algebra structure on $f^*(B)$ is pointwise multiplication on the left, $\Phi_{f^*(B)} : C_0(Y) \longrightarrow ZM(f^*(B))$, $\Phi_{f^*(B)}(h)(g \otimes b) = hg \otimes b$, for $h, g \in C_0(Y)$ and $b \in B$. Note that the fiber $f^*(B)_y$ over $y \in Y$ is isomorphic to $B_{f(y)}$. Indeed, the isomorphism $f^*(B)_y \cong B_{f(y)}$ is induced by the map $C_0(Y) \otimes B \longrightarrow B_{f(y)}$, $f \otimes b \mapsto f(y)\pi_{f(y)}(b)$, whereas the inverse map is induced by $B \to 1 \otimes B \to M(C_0(Y) \otimes B) \to M(C_0(Y) \otimes_{C_0(X)} B) \to M(f^*(B)_y)$.

RKK-theory

Definition 1.4.7 (RKK-groups)**.** Let A and B be two $C_0(X)$-algebras. Add the following additional requirement to Kasparov modules: Kasparov modules $(E, \phi, F) \in \mathbb{E}^i(A, B)$ are to satisfy: $(fa) \cdot e \cdot b = a \cdot e \cdot (fb)$ for any $f \in C_0(X)$, $a \in A$, $b \in B$ and $e \in E$. Then proceed as in Definition 1.2.4 to define $RKK^i(X; A, B)$ by considering such Kasparov modules modulo the homotopy equivalence relation.

Connes-Thom-Kasparov isomorphism

Theorem 1.4.8. *Let A be a $C_0(X)$-algebra and $\alpha : \mathbb{R}^n \longrightarrow Aut(A)$ a fibrewise action. There exists an invertible element*

$$\boldsymbol{t}_\alpha \in RKK^n(X; A, A \rtimes_\alpha \mathbb{R}^n).$$

Hence A and $A \rtimes_\alpha \mathbb{R}^n$ are RKK-equivalent with dimension shift $n\,(mod\,2)$.

RKK-fibrations

Recall that Δ^p denotes the standard p-simplex.

Definition 1.4.9. A $C_0(X)$-algebra B is called a *KK-fibration* if for every positive integer p, every continuous map $f : \Delta^p \longrightarrow X$ and every element $v \in \Delta^p$ the evaluation map

$$q_v : f^*(B) \longrightarrow B_{f(v)}$$

is a KK-equivalence.

Definition 1.4.10. A $C_0(X)$-algebra B is called an *RKK-fibration* if for every positive integer p, every continuous map $f : \Delta^p \longrightarrow X$ and every element $v \in \Delta^p$, $f^*(B)$ is $RKK(\Delta^p; \cdot, \cdot)$-equivalent to $C(\Delta^p, B_{f(v)})$.

Remark 1.4.11. Given a C*-algebra A, the canonical $C_0(X)$-algebra $B = C_0(X)\otimes A$ is an RKK-fibration. Indeed, given $f : \Delta^p \longrightarrow X$ and $v \in \Delta^p$, the pullback

$$f^*(B) = C(\Delta^p) \otimes_{C_0(X)} C_0(X) \otimes A$$

is $C(\Delta^p)$-linearly *-isomorphic to $C(\Delta^p, B_{f(v)}) = C(\Delta^p) \otimes B_{f(v)} = C(\Delta^p) \otimes A$ by the map

$$h \otimes g \otimes a \mapsto \Phi_{C(\Delta^p)}(g)h \otimes a = f^*(g)h \otimes a,$$

where $h \in C(\Delta^p)$, $g \in C_0(X)$ and $a \in A$. This implies the required $RKK(\Delta^p;\cdot,\cdot)$-equivalence.

Note also that the property of being an RKK-fibration is preserved under RKK-equivalence.

The following observation ([22, Remark 1.4]) will be useful.

Lemma 1.4.12. *An RKK-fibration is a KK-fibration.*

Proof. Suppose B is an RKK-fibration, let $f : \Delta^p \longrightarrow X$ and $v \in \Delta^p$. Concisely put, we get the following comutative diagram in the KK category in which all arrows but the right vertical arrow are already known to be isomorphisms

$$\begin{tikzcd} C(\Delta^p, B_{f(v)}) \arrow[r, "\mathbf{r}"] \arrow[d, "ev_v"'] & f^*(B) \arrow[d, "q_v"] \\ B_{f(v)} \arrow[r, "r(v)"'] & B_{f(v)} \end{tikzcd}$$

It follows immediately from the commutative diagram that the right vertical arrow q_v must be an isomorphism as well. $\square$

1.5 C*-correspondences and Cuntz-Pimsner algebras

We collect the basic concepts regarding Cuntz-Pimsner algebras following mainly [32] (see also [42]).

Definition 1.5.1. Given a C*-algebra A, we say X is a C*-correspondence over A when X is a (right) Hilbert A-module with a given *-homomorphism $\varphi_X : A \longrightarrow \mathcal{L}(X)$ called the *left action*. We write (X, A) for a C*-correspondence.

Any C*-algebra A is a C*-correspondence over itself, by letting $\varphi_A : A \longrightarrow \mathcal{L}(A)$ be the left multiplication operator. This is called the identity correspondence.

Given two C*-correspondences (X, A) and (Y, A), we can define a correspondence $X\otimes Y$ over A by considering the internal tensor product Hilbert C*-module $X\otimes_{\varphi_Y} Y$ and defining $\varphi_{X\otimes Y} : A \longrightarrow \mathcal{L}(X \otimes_{\varphi_Y} Y)$, $\varphi_{X\otimes Y}(a)(\xi \otimes \eta) = \varphi_X(a)\xi \otimes \eta$, for $a \in A$, $\xi \in X$ and $\eta \in Y$. This gives the C*-correspondence $(X \otimes Y, A)$.

Given a C*-correspondence (X, A) and $n \in \mathbb{N}$, we denote $X^{\otimes 0} = A$, $X^{\otimes 1} = X$ and $X^{\otimes(n+1)} = X \otimes X^{\otimes n}$.

Definition 1.5.2. A *representation* of a C*-correspondence (X, A) on a C*-algebra B is a pair (π, t) consisting of a *-homomorphism $\pi : A \longrightarrow B$ and a linear map $t : X \longrightarrow B$ such that

(i) $t(\xi)^*t(\eta) = \pi(\langle\xi,\eta\rangle)$, for $\xi,\eta \in X$,

(ii) $\pi(a)t(\xi) = t(\varphi_X(a)\xi)$ for $a \in A$, $\xi \in X$.

A representation (π, t) is said to be injective if the *-homomorphism π is injective. The C*-algebra $C^*(\pi, t)$ is by definition the C*-algebra generated by the images $\pi(A)$ and $t(X)$ in B. The universal representation (which can be obtained by taking the direct sum of sufficiently many representations) is denoted $(\overline{\pi}_X, \overline{t}_X)$. We define $\mathcal{T}_X = C^*(\overline{\pi}_X, \overline{t}_X)$.

Given a representation (π, t) on B of a C*-correspondence (X, A), one defines a *-homomorphism $\psi_t : \mathcal{K}(X) \longrightarrow B$ by $\psi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*$ for $\xi, \eta \in X$.

For every representation (π, t) there is by universality a natural surjection $\rho : \mathcal{T}_X \longrightarrow C^*(\pi, t)$ with $\pi = \rho \circ \overline{\pi}_X$ and $t = \rho \circ \overline{t}_X$.

Let (X, A) be a C*-correspondence. We define an ideal $J_X \subseteq A$ by

$$J_X = \varphi_X^{-1}(\mathcal{K}(X)) \cap (ker\, \varphi_X)^{\perp} = \{a \in A : \varphi_X(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in ker \varphi_X\}.$$

A representation (π, t) is said to be *covariant* if $\pi(a) = \psi_t(\varphi_X(a))$ for all $a \in J_X$.

Definition 1.5.3. Let (X, A) be a C*-correspondence and denote by (π_X, t_X) the universal covariant representation. Define $\mathcal{O}_X = C^*(\pi_X, t_X)$.

Recall that given a C*-correspondence (Y, B), its multiplier bimodule $M(Y)$ is by definition $M(Y) = \mathcal{L}_B(B, Y)$, which together with $M(B)$ naturally constitutes the C*-correspondence $(M(Y), M(B))$.

Definition 1.5.4. Let (X, A) and (Y, B) be C*-correspondences. A correspondence homomorphism $(\psi, \pi) : (X, A) \longrightarrow (M(Y), M(B))$ consists of a *-homomorphism $\pi : A \longrightarrow M(B)$ and linear map $\psi : X \longrightarrow M(Y)$ preserving the correspondence operations.

A correspondence homomorphism $(\psi, \pi) : (X, A) \longrightarrow (M(Y), M(B))$ is called *Cuntz-Pimsner covariant* if $\psi(X) \subset M_B(Y)$, π is nondegenerate, $\pi(J_X) \subset M(B; J_Y)$ and $\phi_B \circ \pi = \psi^{(1)} \circ \phi_A$, where ϕ_A and ϕ_B denote the respective left actions, and $\psi^{(1)}$ denotes the induced map $\psi^{(1)} : \mathcal{K}(X) \longrightarrow \mathcal{L}(Y)$, $\psi^{(1)}(\theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*$.

Let $n \in \mathbb{N}$ and $X^{\otimes n}$ be the tensor product C*-correspondence, denote by $\varphi_n : A \longrightarrow \mathcal{L}(X^{\otimes n})$ the natural left action. For each $\xi \in X^{\otimes n}$ and $m \in \mathbb{N}$ define the operator $\tau_m^n(\xi) \in \mathcal{L}(X^{\otimes m}, X^{\otimes(n+m)})$, $\tau_m^n(\xi)\eta = \xi \otimes \eta$. The Fock space is the Hilbert A-module $\mathcal{F}(X) = \sum_{n \geqslant 0} X^{\otimes n}$. The Fock representation is given by the pair $(\varphi_\infty, \tau_\infty)$ where $\varphi_\infty : A \longrightarrow \mathcal{L}(\mathcal{F}(X))$, $\varphi_\infty(a) = \sum_m \varphi_m(a)$, and $\tau_\infty : X \longrightarrow \mathcal{L}(\mathcal{F}(X))$, $\tau_\infty(\xi) = \sum_m \tau_m^1(\xi)$. The set $\mathcal{F}(X)J_X$ is a Hilbert J_X-module and $\mathcal{K}(\mathcal{F}(X)J_X)$ is an ideal in $\mathcal{L}(\mathcal{F}(X))$, let (φ, τ) denote the composition of $(\varphi_\infty, \tau_\infty)$ with the quotient map $\mathcal{L}(\mathcal{F}(X)) \longrightarrow \mathcal{L}(\mathcal{F}(X))/\mathcal{K}(\mathcal{F}(X)J_X)$. It follows that (φ, τ) is a covariant representation of X. It is an important fact that $(\varphi_\infty, \tau_\infty)$ and (φ, τ) are injective representations.

For a representation (π, t) of X we set $I'_{(\pi,t)} = \{a \in A : \pi(a) \in \psi_t(\mathcal{K}(X))\}$. A representation (π, t) is said to admit a *gauge action* if for each $z \in \mathbb{T}$ there exists a *-homomorphism $\beta_z : C^*(\pi, t) \longrightarrow C^*(\pi, t)$ such that $\beta_z(\pi(a)) = \pi(a)$ and $\beta_z(t(\xi)) = zt(\xi)$ for all $a \in A$ and $\xi \in X$. The following are the gauge invariant uniqueness theorems, which are quite useful for providing explicit realizations of the algebras.

Theorem 1.5.5. *For a representation (π, t) of X the surjection $\rho : \mathcal{T}_X \longrightarrow C^*(\pi, t)$ is an isomorphism if and only if (π, t) satisfies $I'_{(\pi,t)} = 0$ and admits a gauge action.*

Theorem 1.5.6. *For a covariant representation (π, t) of a C^*-correspondence X, the *-homomorphism $\rho : \mathcal{O}_X \longrightarrow C^*(\pi, t)$ is an isomorphism if and only if (π, t) is injective and admits a gauge action.*

Corollary 1.5.7. *The representations $(\varphi_\infty, \tau_\infty)$ and (φ, τ) are universal, that is, we have natural isomorphisms $C^*(\varphi_\infty, \tau_\infty) \cong \mathcal{T}_X$ and $C^*(\varphi, \tau) \cong \mathcal{O}_X$.*

Crossed products of C*-correspondences by coactions

This material is mainly from the recent paper [29].

Definition 1.5.8. A coaction of a locally compact group G on a C*-correspondence (X, A) is a nondegenerate correspondence homomorphism

$$(\sigma, \delta) : (X, A) \longrightarrow (M(X \otimes C^*(G)), M(A \otimes C^*(G)))$$

such that

(i) $\delta : A \longrightarrow M(A \otimes C^*(G))$ is a coaction,

(ii) $(\sigma \otimes 1)\sigma = (1 \otimes \delta_G)\sigma$, (coaction identity)

(iii) $\overline{(1 \otimes C^*(G))\sigma(X)} = X \otimes C^*(G)$, (nondegeneracy)

Let $j_X = (1 \otimes \lambda)\sigma : X \longrightarrow M(X \otimes K(L^2(G)))$ and $j_G^A = 1 \otimes M : C_0(G) \longrightarrow M(X \otimes K(L^2(G)))$.

The crossed product of X by the coaction of G is defined as

$$X \rtimes_\sigma G = \overline{span}\{j_X(X) j_G^A(C_0(G))\} \subset M(X \otimes K(L^2(G))).$$

We get the crossed product C*-correspondence $(X \rtimes_\sigma G, A \rtimes_\delta G)$, and $(j_X, j_A) : (X, A) \longrightarrow (M(X \rtimes_\sigma G), M(A \rtimes_\delta G))$ is a correspondence homomorphism.

Let (σ, δ) be a coaction of G on a correspondence (X, A). Then the canonical correspondence homomorphism $(j_X, j_A) : (X, A) \longrightarrow (M(X \rtimes_\sigma G), M(A \rtimes_\delta G))$ is Cuntz-Pimsner covariant if and only if

$$j_A(J_X) \subset M(A \rtimes_\delta G; J_{X \rtimes_\sigma G}).$$

Moreover, if (σ, δ) is Cuntz-Pimsner covariant, then there is a unique coaction $\zeta : \mathcal{O}_X \longrightarrow M(\mathcal{O}_X \otimes C^*(G))$ such that $(k_X \otimes 1, k_A \otimes 1) \circ (\sigma, \delta) = \zeta \circ (k_X, k_A)$.

Theorem 1.5.9. *Let (σ, δ) be a Cuntz-Pimsner covariant coaction of G on a correspondence (X, A) and let ζ be the associated coaction on $\mathcal{O}_X$. If the canonical correspondence homomorphism (j_X, j_A) is Cuntz-Pimsner covariant, then*

$$\mathcal{O}_X \rtimes_\zeta G \cong \mathcal{O}_{X \rtimes_\sigma G}.$$

There is an analogous development for C*-correspondences associated to crossed products by actions ([25]), but we omit it from the presentation as we do not make use of it later. However, the results we will consider, namely deformation of the C*-correspondence and the resulting deformed Cuntz-Pimsner algebra, work analogously for C*-correspondences of crossed products by actions.

Part II

Deformation and K-theory

Chapter 2

Cocycle deformation

We develop an approach to deformation of operator algebras that generalizes Kasprzak's approach to Rieffel deformation. Given a coaction δ of a locally compact group G on a C*-algebra A and a $\mathbb{T}$-valued Borel 2-cocycle ω on G, we define a deformation A_ω of A. Among other properties of A_ω we show that the stabilization $A_\omega \otimes K(L^2(G))$ is canonically isomorphic to the twisted crossed product $A \rtimes_\delta \widehat{G} \rtimes_{\hat{\delta},\omega} G$. We also show an invariance result in K-theory for the deformation.

2.1 Deformation of algebras

Let G be a second countable locally compact group. Denote by $Z^2(G;\mathbb{T})$ the set of $\mathbb{T}$-valued Borel 2-cocycles on G, so $\omega \in Z^2(G;\mathbb{T})$ is a Borel function $G \times G \to \mathbb{T}$ such that

$$\omega(g,h)\omega(gh,k) = \omega(g,hk)\omega(h,k).$$

For every cocycle ω consider also the cocycles $\tilde{\omega}$ and $\bar{\omega}$ defined by

$$\tilde{\omega}(g,h) = \omega(h^{-1},g^{-1}) \quad \text{and} \quad \bar{\omega}(g,h) = \overline{\omega(g,h)}.$$

Define operators λ^ω_g and $\rho^{\tilde{\omega}}_g$ on $L^2(G)$ by [1]

$$\lambda^\omega_g = \tilde{\omega}(g^{-1},\cdot)\lambda_g, \quad \rho^{\tilde{\omega}}_g = \tilde{\omega}(\cdot,g)\rho_g.$$

Then

$$\lambda^\omega_g\lambda^\omega_h = \omega(g,h)\lambda^\omega_{gh}, \quad \rho^{\tilde{\omega}}_g\rho^{\tilde{\omega}}_h = \tilde{\omega}(g,h)\rho^{\tilde{\omega}}_{gh} \;\; \text{and} \;\; [\lambda^\omega_g,\rho^{\tilde{\omega}}_h] = 0 \;\; \text{for all} \;\; g,h \in G.$$

Fix now a cocycle $\omega \in Z^2(G;\mathbb{T})$ and consider a coaction δ of G on a C*-algebra A. Assume first that the cocycle ω is continuous. In this case the functions $\tilde{\omega}(\cdot,g)$ belong to the multiplier algebra of $C_0(G)$, so we can define a new twisted dual action

[1] The operators λ^ω_g and $\rho^{\tilde{\omega}}_g$ are more commonly defined by $\lambda^\omega_g = \lambda_g\omega(g,\cdot) = \omega(g,g^{-1}\cdot)\lambda_g$ and $\rho^{\tilde{\omega}}_g = \rho_g\omega(\cdot,g^{-1}) = \omega(\cdot g,g^{-1})\rho_g$. With our definition some of the formulas will look better. If the cocycle ω satisfies $\omega(g,e) = \omega(e,g) = \omega(g,g^{-1}) = 1$ for all $g \in G$, then the two definitions coincide, that is to say $\omega(h^{-1},g) = \omega(g,g^{-1}h)$, which follows by applying the cocycle identity for ω to the triple $(h^{-1},g,g^{-1}h)$. Any cocycle is cohomologous to a cocycle satisfying the above normalization conditions, so in principle we could consider only such cocycles.

$\hat\delta^\omega$ on $A\rtimes_\delta \widehat{G}$ by letting $\hat\delta^\omega_g=\operatorname{Ad}(1\otimes\rho^{\tilde\omega}_g)$. In other words, if we consider $\tilde\omega$ as a multiplier of $C_0(G)\otimes C_0(G)$, then

$$\hat\delta^\omega(x)=\tilde\omega_{23}\hat\delta(x)\tilde\omega^*_{23}=\tilde\omega_{23}V_{23}(x\otimes 1)V^*_{23}\tilde\omega^*_{23}\in M(A\otimes K\otimes K).$$

For $f\in C_0(G)$ we obviously have $\hat\delta^\omega(1\otimes f)=\hat\delta(1\otimes f)=1\otimes\Delta(f)$. By the Landstad-type duality result of Quigg and Vaes (see subsection on Duality results of section 1.1 of chapter 1) it follows that $\hat\delta^\omega$ is the dual action on a crossed product $A_\omega\rtimes_{\delta^\omega}\widehat{G}$ for some C*-subalgebra $A_\omega\subset M(A\rtimes_\delta\widehat{G})\subset M(A\otimes K)$ and a coaction δ^ω of G, and this subalgebra is defined using slice maps applied to the image of $A\rtimes_\delta\widehat{G}$ under the homomorphism

$$\eta^\omega\colon A\rtimes_\delta\widehat{G}\to M(A\otimes K\otimes K),\quad \eta^\omega(x)=W_{23}\tilde\omega_{23}\hat\delta(x)\tilde\omega^*_{23}W^*_{23}.$$

If the cocycle ω is only assumed to be Borel, it is not clear whether the action $\hat\delta^\omega$ is well-defined. Nevertheless, the homomorphism $\eta^\omega\colon A\rtimes_\delta G\to M(A\otimes K\otimes K)$ defined above still makes sense. Therefore we can give the following definition.

Definition 2.1.1. The ω-deformation of a C*-algebra A equipped with a coaction δ of G is the C*-subalgebra $A_\omega\subset M(A\otimes K)$ generated by all elements of the form

$$(\iota\otimes\iota\otimes\varphi)\eta^\omega\delta(a)=(\iota\otimes\iota\otimes\varphi)\operatorname{Ad}(W_{23}\tilde\omega_{23})(\delta(a)\otimes 1),$$

where $a\in A$ and $\varphi\in K^*$.

This generalizes the approach of Kasprzak. In case we want to stress that the deformation is defined using the coaction δ, we will write $A_{\delta,\omega}$ instead of A_ω.

Note that if we considered elements of the form $(\iota\otimes\iota\otimes\varphi)\eta^\omega(x)$ for all $x\in A\rtimes_\delta\widehat{G}$, this would not change the algebra A_ω, since $\eta^\omega(1\otimes f)=1\otimes 1\otimes f$.

In order to get an idea about the structure of A_ω consider the C*-algebra $C^*_r(G,\omega)$ generated by operators of the form

$$\lambda^\omega_f=\int_G f(g)\lambda^\omega_g dg,\quad f\in L^1(G).$$

When necessary we denote by λ^ω the identity representation of $C^*_r(G,\omega)$ on $L^2(G)$. A simple computation shows that

$$W\tilde\omega(\lambda_g\otimes 1)\tilde\omega^*W^*=\lambda^\omega_g\otimes\lambda^{\bar\omega}_g. \tag{2.1.1}$$

The map $g\mapsto\lambda^\omega_g\otimes\lambda^{\bar\omega}_g$ therefore defines a representation of G on $L^2(G\times G)$ that is quasi-equivalent to the regular representation, so it defines a representation of $C^*_r(G)$. Denote this representation by $\lambda^\omega\boxtimes\lambda^{\bar\omega}$. We can then write

$$\eta^\omega\delta(a)=(\iota\otimes(\lambda^\omega\boxtimes\lambda^{\bar\omega}))\delta(a)\ \text{ for }\ a\in A.$$

Since the image of $C^*_r(G)$ under $\lambda^\omega\boxtimes\lambda^{\bar\omega}$ is contained in $M(C^*_r(G,\omega)\otimes C^*_r(G,\bar\omega))$, we see in particular that $A_\omega\subset M(A\otimes C^*_r(G,\omega))$.

Example 2.1.2. Assume the group G is discrete. Denote by $A_g\subset A$ the spectral subspace corresponding to $g\in G$, so A_g consists of all elements $a\in A$ such that $\delta(a)=a\otimes\lambda_g$. The spaces A_g, $g\in G$, span a dense $*$-subalgebra $\mathcal{A}\subset A$. By (2.1.1), if $a \in A_g$ then $\eta^\omega\delta(a) = a\otimes\lambda^\omega_g\otimes\lambda^{\bar\omega}_g$. Thus the linear span of elements $(\iota\otimes\iota\otimes\varphi)\eta^\omega\delta(a)$, with $a \in \mathcal{A}$ and $\varphi \in K^*$, coincides with the linear span $\mathcal{A}_\omega$ of elements $a \otimes \lambda^\omega_g$, with $a \in A_g$ and $g \in G$. The space $\mathcal{A}_\omega$ is already a $*$-algebra and A_ω is the closure of $\mathcal{A}_\omega$ in $A \otimes C^*_r(G,\omega)$. In particular, we see that for discrete groups our definition of ω-deformation is equivalent to that of Yamashita, see [62, Proposition 2].

The following theorem is the first principal result of this section.

Theorem 2.1.3. *The C*-algebra $A_\omega \subset M(A \otimes K)$ coincides with the norm closure of the linear span of elements of the form $(\iota\otimes\iota\otimes\varphi)\eta^\omega\delta(a)$, where $a \in A$ and $\varphi \in K^*$.*

While proving this theorem we will simultaneously obtain a description of $A_\omega\otimes K$. We need to introduce more notation in order to formulate the result.

In addition to λ^ω we have another equivalent representation ρ^ω of $C^*_r(G,\omega)$ on $L^2(G)$ that maps $\lambda^\omega_g \in M(C^*_r(G,\omega))$ into ρ^ω_g.

Given an action α of G on a C*-algebra B, the reduced twisted crossed product is defined by

$$B \rtimes_{\alpha,\omega} G = \overline{\alpha(B)(1 \otimes \rho^\omega(C^*_r(G,\omega)))} \subset M(B \otimes K).$$

The reduced twisted crossed product has a dual coaction, which we again denote by $\hat\alpha$, defined by

$$\hat\alpha(x) = W^*_{23}(x \otimes 1)W_{23}, \quad \text{so} \quad \hat\alpha(\alpha(b)) = \alpha(b) \otimes 1, \quad \hat\alpha(1 \otimes \rho^\omega_g) = 1 \otimes \rho^\omega_g \otimes \lambda_g.$$

The last ingredient that we need is the well-known fact that the cocycles $\tilde\omega$ and $\bar\omega$ are cohomologous. Explicitly,

$$\tilde\omega(g,h) = \bar\omega(g,h)v(g)v(h)v(gh)^{-1}, \quad \text{where} \quad v(g) = \omega(g^{-1},g)\omega(e,e).$$

This follows from the cocycle identities

$$\omega(h^{-1},g^{-1})\omega(h^{-1}g^{-1},gh) = \omega(h^{-1},h)\omega(g^{-1},gh), \;\; \omega(g^{-1},gh)\omega(g,h) = \omega(g^{-1},g)\omega(e,h);$$

recall also that $\omega(e,h) = \omega(e,e)$ for all h, which follows from the cocycle identity applied to (e,e,h).

We can now formulate our second principal result.

Theorem 2.1.4. *Put $u(g) = \overline{\omega(g^{-1},g)\omega(e,e)}$. Then the map*

$$\mathrm{Ad}((1 \otimes W\tilde\omega)(1 \otimes 1 \otimes u))\colon A \rtimes_\delta \widehat{G} \rtimes_{\hat\delta,\omega} G \to M(A \otimes K \otimes K)$$

defines an isomorphism $A \rtimes_\delta \widehat{G} \rtimes_{\hat\delta,\omega} G \cong A_\omega \otimes K$.

For discrete groups the fact that the C*-algebras A_ω and $A \rtimes_\delta \widehat{G} \rtimes_{\hat\delta,\omega} G$ are strongly Morita equivalent was observed by Yamashita [62, Corollary 15].

Proof of Theorems 2.1.3 and 2.1.4. Denote by θ the map in the formulation of Theorem 2.1.4. In order to compute its image, observe first that since

$$\bar{\tilde\omega}(h,g) = \omega(h,g)u(h)u(g)u(hg)^{-1},$$

we have
$$u\rho_g^\omega u^* = \overline{u(g)}\rho_g^{\bar{\bar{\omega}}}.$$
Next, it is straightforward to check that $W\tilde\omega$ commutes with $1\otimes\rho_g^{\bar{\bar{\omega}}}$. We thus see that θ acts as
$$\delta(a)\otimes 1\mapsto \eta^\omega\delta(a),\ \ 1\otimes\Delta(f)\mapsto 1\otimes 1\otimes f,\ \ 1\otimes 1\otimes\rho_g^\omega\mapsto 1\otimes 1\otimes u\rho_g^\omega u^*.$$
In particular, we see that the image of the C*-subalgebra
$$\overline{(1\otimes\Delta(C_0(G)))(1\otimes 1\otimes\rho^\omega(C_r^*(G,\omega)))}\cong C_0(G)\rtimes_{\operatorname{Ad}\rho,\omega}G$$
of $M(A\rtimes_\delta\hat G\rtimes_{\hat\delta,\omega}G)$ is
$$1\otimes 1\otimes\overline{uC_0(G)C_r^*(G,\omega)u^*} = 1\otimes 1\otimes K.$$
Therefore $1\otimes 1\otimes K$ is a nondegenerate C*-subalgebra of $M(\theta(A\rtimes_\delta\hat G\rtimes_{\hat\delta,\omega}G))\subset M(A\otimes K\otimes K)$. It follows that there exists a uniquely defined C*-subalgebra $A_1\subset M(A\otimes K)$ such that
$$\theta(A\rtimes_\delta\hat G\rtimes_{\hat\delta,\omega}G) = A_1\otimes K.$$
By definition of crossed products and the above computation of θ we then have
$$A_1\otimes K = \overline{\eta^\omega\delta(A)(1\otimes 1\otimes K)}.$$
Applying the slice maps $\iota\otimes\iota\otimes\varphi$ we conclude that the closed linear span of elements of the form $(\iota\otimes\iota\otimes\varphi)\eta^\omega\delta(a)$ coincides with the C*-algebra A_1. This finishes the proof of both theorems. $\square$

Theorem 2.1.4 essentially reduces the study of ω-deformations to that of (twisted) crossed products. As a simple illustration let us prove the following result that refines and generalizes [62, Proposition 14].

Proposition 2.1.5. *Assume we are given two exterior equivalent coactions δ and δ_X of G on a C*-algebra A. Then $A_{\delta,\omega}\otimes K\cong A_{\delta_X,\omega}\otimes K$.*

Proof. Since δ and δ_X are exterior equivalent, we have $(A\rtimes_\delta\hat G,\hat\delta)\cong(A\rtimes_{\delta_X}\hat G,\hat\delta_X)$, and hence $A\rtimes_\delta\hat G\rtimes_{\hat\delta,\omega}G\cong A\rtimes_{\delta_X}\hat G\rtimes_{\hat\delta_X,\omega}G$. $\square$

Note that for continuous cocycles this result is also a consequence of the following useful fact combined with the Takesaki-Takai duality.

Proposition 2.1.6. *If the cocycle ω is continuous, then any two exterior equivalent coactions have exterior equivalent twisted dual actions. More precisely, assume $X\in M(A\otimes C_r^*(G))$ is a 1-cocycle for a coaction δ of G on A. Then the element $U = X_{12}\tilde\omega_{23}X_{12}^*\tilde\omega_{23}^*\in M(A\otimes K\otimes C_0(G))$ is a 1-cocycle for the action $\hat\delta_X^\omega$ of G on $A\rtimes_{\delta_X}\hat G$, and the isomorphism $\operatorname{Ad}X\colon A\rtimes_\delta\hat G\to A\rtimes_{\delta_X}\hat G$ intertwines $\hat\delta^\omega$ with $(\hat\delta_X^\omega)_U$.*

Proof. Denote by Ψ the isomorphism $\operatorname{Ad}X\colon A\rtimes_\delta\hat G\to A\rtimes_{\delta_X}\hat G$ and put
$$Y = 1\otimes\tilde\omega\in M(1\otimes C_0(G)\otimes C_0(G))\subset M((A\rtimes_\delta\hat G)\otimes C_0(G))\cap M((A\rtimes_{\delta_X}\hat G)\otimes C_0(G)).$$

Then $U = (\Psi \otimes \iota)(Y)Y^* \in M((A \rtimes_{\delta_X} \widehat{G}) \otimes C_0(G))$. In order to show that U is a 1-cocycle for $\hat{\delta}^\omega_X$, observe first that

$$(Y \otimes 1)(\hat{\delta}_X \otimes \iota)(Y) = (\iota \otimes \iota \otimes \Delta)(Y)\tilde{\omega}_{34}, \tag{2.1.2}$$

which is simply the cocycle identity for $\tilde{\omega}$. We also have the same identity for $\hat{\delta}$. Furthermore, since Ψ intertwines $\hat{\delta}$ with $\hat{\delta}_X$, we also get

$$((\Psi \otimes \iota)(Y) \otimes 1)(\hat{\delta}_X \otimes \iota)(\Psi \otimes \iota)(Y) = (\iota \otimes \iota \otimes \Delta)(\Psi \otimes \iota)(Y)\tilde{\omega}_{34}.$$

Multiplying this identity by the adjoint of (2.1.2) we obtain

$$((\Psi \otimes \iota)(Y) \otimes 1)(\hat{\delta}_X \otimes \iota)(U)(Y^* \otimes 1) = (\iota \otimes \iota \otimes \Delta)(U).$$

Since $\hat{\delta}^\omega_X = Y\hat{\delta}_X(\cdot)Y^*$, this is exactly the cocycle identity

$$(U \otimes 1)(\hat{\delta}_X \otimes \iota)(U) = (\iota \otimes \iota \otimes \Delta)(U).$$

Since $\hat{\delta}^\omega = Y\hat{\delta}(\cdot)Y^*$, $\hat{\delta}^\omega_X = Y\hat{\delta}_X(\cdot)Y^*$ and Ψ intertwines $\hat{\delta}$ with $\hat{\delta}_X$, we immediately see that Ψ intertwines $\hat{\delta}^\omega$ with $(\Psi \otimes \iota)(Y)\hat{\delta}_X(\cdot)(\Psi \otimes \iota)(Y)^* = U\hat{\delta}^\omega_X(\cdot)U^*$. □

We finish the section with the following simple observation.

Proposition 2.1.7. *Assume $\omega_1, \omega_2 \in Z^2(G; \mathbb{T})$ are cohomologous cocycles. Then $A_{\omega_1} \cong A_{\omega_2}$.*

Proof. By assumption there exists a Borel function $v\colon G \to \mathbb{T}$ such that

$$\tilde{\omega}_1(g, h) = \tilde{\omega}_2(g, h)v(g)v(h)v(gh)^{-1},$$

that is, $\tilde{\omega}_1 = \tilde{\omega}_2(v \otimes v)\Delta(v)^*$. Note that then $\lambda^{\omega_1}_g = v(g^{-1})v\lambda^{\omega_2}_g v^*$. Using that $W\Delta(v)W^* = 1 \otimes v$ and that W commutes with $v \otimes 1$, for any operator x on $L^2(G)$ we get

$$W\tilde{\omega}_1(x \otimes 1)\tilde{\omega}_1^* W^* = (v \otimes v^*)W\tilde{\omega}_2(x \otimes 1)\tilde{\omega}_2^* W^*(v^* \otimes v).$$

This shows that

$$\eta^{\omega_1} = \mathrm{Ad}(1 \otimes v \otimes v^*)\eta^{\omega_2},$$

which in turn gives $A_{\omega_1} = \mathrm{Ad}(1 \otimes v)(A_{\omega_2})$. □

2.2 Canonical and dual coactions

By the Landstad-type result of Quigg and Vaes the twisted dual action $\hat{\delta}^\omega$, when it is defined, is dual to some coaction. The action $\hat{\delta}^\omega$ is apparently not always well-defined on $A \rtimes_\delta \widehat{G}$. Nevertheless the new coaction on A_ω always makes sense.

Theorem 2.2.1. *For any cocycle $\omega \in Z^2(G; \mathbb{T})$ and a coaction δ of G on a C^*-algebra A we have:*

(i) *the formula $\delta^\omega(x) = W_{23}(x \otimes 1)W_{23}^*$ defines a coaction of G on A_ω;*

(ii) *if the twisted dual action $\hat{\delta}^\omega$ is well-defined on $A \rtimes_\delta G$, then $A \rtimes_\delta \widehat{G} = \overline{A_\omega(1 \otimes C_0(G))}$ and the map $\eta^\omega\colon A \rtimes_\delta \widehat{G} \to M(A \otimes K \otimes K)$ gives an isomorphism $A \rtimes_\delta \widehat{G} \cong A_\omega \rtimes_{\delta^\omega} \widehat{G}$ that intertwines the twisted dual action $\hat{\delta}^\omega$ on $A \rtimes_\delta \widehat{G}$ with the dual action to δ^ω on $A_\omega \rtimes_{\delta^\omega} \widehat{G}$.*

Proof. (i) We repeat the computations of Vaes in the proof [57, Theorem 6.7]. Since

$$W_{13}W_{12} = (\iota \otimes \hat{\Delta})(W) = W_{23}W_{12}W_{23}^*,$$

for $x = (\iota \otimes \iota \otimes \varphi)\eta^\omega(y)$, $y \in A \rtimes_\delta \widehat{G}$, we have

$$\begin{aligned}
\delta^\omega(x) &= (\iota \otimes \iota \otimes \varphi \otimes \iota)(W_{24}(\eta^\omega(y) \otimes 1)W_{24}^*) \\
&= (\iota \otimes \iota \otimes \varphi \otimes \iota)(W_{24}W_{23}\tilde{\omega}_{23}(\hat{\delta}(y) \otimes 1)\tilde{\omega}_{23}^* W_{23}^* W_{24}^*) \\
&= (\iota \otimes \iota \otimes \varphi \otimes \iota)(W_{34}W_{23}W_{34}^*\tilde{\omega}_{23}(\hat{\delta}(y) \otimes 1)\tilde{\omega}_{23}^* W_{34} W_{23}^* W_{34}^*) \\
&= (\iota \otimes \iota \otimes \varphi \otimes \iota)(W_{34}(\eta^\omega(y) \otimes 1)W_{34}^*).
\end{aligned}$$

From this one can easily see that the closure of $\delta^\omega(A_\omega)(1 \otimes 1 \otimes C_r^*(G))$ coincides with $A_\omega \otimes C_r^*(G)$, because $\overline{(K \otimes 1)W(1 \otimes C_r^*(G))} = K \otimes C_r^*(G)$ and $W^*(K \otimes C_r^*(G)) = K \otimes C_r^*(G)$. Since $1 \otimes W$ is a 1-cocycle for the trivial coaction on $A \otimes K$ (so $(\iota \otimes \hat{\Delta})(W) = W_{12}W_{13}$), the identity $(\iota \otimes \hat{\Delta})\delta^\omega = (\delta^\omega \otimes \iota)\delta^\omega$ follows.

(ii) This is [57, Theorem 6.7] applied to the action $\hat{\delta}^\omega$. □

The twisted dual action is well-defined for continuous cocycles, but as the following result shows it can also be well-defined even if the cocycle is only Borel.

Proposition 2.2.2. *If δ is a dual coaction, then the twisted dual action $\hat{\delta}^\omega$ of G on $A \rtimes_\delta \widehat{G}$ is well-defined for any $\omega \in Z^2(G; \mathbb{T})$.*

Proof. By assumption we have $A = B \rtimes_\alpha G$ and $\delta = \hat{\alpha}$ for some B and α. Then $A \rtimes_\delta \widehat{G} = B \rtimes_\alpha G \rtimes_{\hat{\alpha}} \widehat{G}$ is the closure of

$$(\alpha(B) \otimes 1)(1 \otimes (\rho \otimes \lambda)\hat{\Delta}(C_r^*(G)))(1 \otimes 1 \otimes C_0(G)) \subset M(B \otimes K \otimes K).$$

We have to check that the inner automorphisms $\mathrm{Ad}(1 \otimes 1 \otimes \rho_g^{\tilde{\omega}})$ of $B \otimes K \otimes K$ define a (continuous) action of G on this closure. Since these automorphisms act trivially on $\alpha(B) \otimes 1$, we just have to check that the automorphisms $\mathrm{Ad}(1 \otimes \rho_g^{\tilde{\omega}})$ of $K \otimes K$ define an action on the C*-algebra

$$\overline{(\rho \otimes \lambda)\hat{\Delta}(C_r^*(G))(1 \otimes C_0(G))} \cong C_r^*(G) \rtimes \widehat{G}.$$

The operator V commutes with $1 \otimes \tilde{\omega}(\cdot, g)$, and $\mathrm{Ad}\, V^*$ maps the above algebra onto $1 \otimes K$. Hence $\mathrm{Ad}(1 \otimes \tilde{\omega}(\cdot, g))$, and therefore also $\mathrm{Ad}(1 \otimes \rho_g^{\tilde{\omega}})$, is a well-defined automorphism of that algebra. Finally, the continuity of the action holds, since any Borel homomorphism of G into a Polish group, such as the group $\mathrm{Aut}(K)$, is automatically continuous. □

For dual coactions it is, however, straightforward to describe the deformed algebra, see [62, Example 8] for the discrete group case. In order to formulate the result, define a unitary W^ω on $L^2(G \times G)$ by

$$(W^\omega \xi)(g, h) = \tilde{\omega}(g^{-1}, h)\xi(g, g^{-1}h).$$

In other words, if we let $W^*(G, \omega) = C_r^*(G, \omega)''$, then $W^\omega \in L^\infty(G)\bar{\otimes} W^*(G, \omega) = L^\infty(G; W^*(G, \omega))$ and $W^\omega(g) = \lambda_g^\omega$.

Proposition 2.2.3. *Assume α is an action of G on a C*-algebra B. Consider the dual coaction δ on $A = B \rtimes_\alpha G$. Then for any $\omega \in Z^2(G;\mathbb{T})$ the map*

$$B \rtimes_{\alpha,\omega} G \mapsto M(B \otimes K \otimes K), \quad x \mapsto W^{\omega *}_{23}(x \otimes 1)W^{\omega}_{23},$$

defines an isomorphism $(B \rtimes_{\alpha,\omega} G, \hat{\alpha}) \cong (A_\omega, \delta^\omega)$.

Proof. First of all observe that by (2.1.1) we have

$$\eta^\omega(\delta(1 \otimes \rho_g)) = 1 \otimes \rho_g \otimes \lambda^\omega_g \otimes \lambda^{\bar{\omega}}_g.$$

This implies that A_ω is the closed linear span of elements of the form

$$(\delta(b) \otimes 1) \int_G f(g)(1 \otimes \rho_g \otimes \lambda^\omega_g) dg,$$

where $b \in B$ and $f \in L^1(G)$. Using the easily verifiable identity

$$W^{\omega *}(\rho^\omega_g \otimes 1)W^\omega = \rho_g \otimes \lambda^\omega_g,$$

we get the required isomorphism

$$\overline{\alpha(B)(1 \otimes \rho^\omega(C^*_r(G,\omega)))} \to A_\omega, \quad x \mapsto W^{\omega *}_{23}(x \otimes 1)W^\omega_{23}.$$

In order to see that this isomorphism respects the coactions, we just have to check that

$$\delta^\omega(1 \otimes \rho_g \otimes \lambda^\omega_g) = 1 \otimes \rho_g \otimes \lambda^\omega_g \otimes \lambda_g,$$

that is, $W(\lambda^\omega_g \otimes 1)W^* = \lambda^\omega_g \otimes \lambda_g$. But this follows immediately from $W(\lambda_g \otimes 1)W^* = \lambda_g \otimes \lambda_g$, since λ^ω_g is λ_g multiplied by a function that automatically commutes with the first leg of W. □

Consider now an arbitrary coaction δ of G on a C*-algebra A and choose two cocycles $\omega, \nu \in Z^2(G;\mathbb{T})$. Using the coaction δ^ω on A_ω we can define the ν-deformation $(A_\omega)_\nu$ of A_ω.

Proposition 2.2.4. *The map*

$$A_{\omega\nu} \to M(A \otimes K \otimes K), \quad x \mapsto W_{23}\tilde{\nu}^*_{23}(x \otimes 1)\tilde{\nu}_{23}W^*_{23},$$

defines an isomorphism $A_{\omega\nu} \cong (A_\omega)_\nu$. In particular, the map $\eta^\omega\delta \colon A \to M(A \otimes K \otimes K)$ defines an isomorphism $A \cong (A_\omega)_{\bar{\omega}}$.

Proof. For $a \in A$ and $\varphi \in K^*$ consider the element

$$x = (\iota \otimes \iota \otimes \varphi)\eta^\omega\delta(a) = (\iota \otimes \iota \otimes \varphi)(\iota \otimes (\lambda^\omega \boxtimes \lambda^{\bar{\omega}}))\delta(a) \in A_\omega.$$

Then

$$\delta^\omega(x) = W_{23}(x \otimes 1)W^*_{23} = (\iota \otimes \iota \otimes \varphi \otimes \iota)(W_{24}((\iota \otimes (\lambda^\omega \boxtimes \lambda^{\bar{\omega}}))\delta(a) \otimes 1)W^*_{24}).$$

Since $W(\lambda^\omega_g \otimes 1)W^* = \lambda^\omega_g \otimes \lambda_g$, as was already used in the proof of the previous proposition, the above identity can be written as

$$\delta^\omega(x) = (\iota \otimes \iota \otimes \varphi \otimes \iota)(\iota \otimes ((\lambda^\omega \boxtimes \lambda^{\bar{\omega}}) \boxtimes \lambda))\delta(a).$$

It follows that

$$\eta^\nu\delta^\omega(x) = (\iota\otimes\iota\otimes\varphi\otimes\iota\otimes\iota)(\iota\otimes((\lambda^\omega\boxtimes\lambda^{\bar\omega})\boxtimes(\lambda^\nu\boxtimes\lambda^{\bar\nu})))\delta(a).$$

Therefore $(A_\omega)_\nu$ is the closed linear span of elements of the form

$$(\iota\otimes\iota\otimes\varphi\otimes\iota\otimes\psi)(\iota\otimes((\lambda^\omega\boxtimes\lambda^{\bar\omega})\boxtimes(\lambda^\nu\boxtimes\lambda^{\bar\nu})))\delta(a),$$

where $a\in A$ and $\varphi,\psi\in K^*$.

Observe next that

$$W\tilde\nu^*(\lambda^{\omega\nu}_g\otimes 1)\tilde\nu W^* = \lambda^\omega_g\otimes\lambda^\nu_g,$$

which is simply identity (2.1.1) for the cocycle $\bar\nu$ multiplied on the left by

$$\tilde\omega(g^{-1},\cdot)\tilde\nu(g^{-1},\cdot)\otimes 1.$$

It follows that the unitary

$$\Sigma_{23}(\tilde\nu W^*\otimes\tilde\nu^*W^*)\Sigma_{23}\ \text{ on }\ L^2(G)^{\otimes 4},$$

where Σ is the flip, intertwines the representation $(\lambda^\omega\boxtimes\lambda^{\bar\omega})\boxtimes(\lambda^\nu\boxtimes\lambda^{\bar\nu})$ of $C^*_r(G)$ with the representation $(\lambda^{\omega\nu}\boxtimes\lambda^{\bar\omega\bar\nu})\otimes 1\otimes 1$. Furthermore, for any $y\in C^*_r(G)$ we have

$$(\operatorname{Ad}\tilde\nu W^*)(\iota\otimes\varphi\otimes\iota\otimes\psi)((\lambda^\omega\boxtimes\lambda^{\bar\omega})\boxtimes(\lambda^\nu\boxtimes\lambda^{\bar\nu}))(y) = \varpi_{24}((\lambda^{\omega\nu}\boxtimes\lambda^{\bar\omega\bar\nu})(y)\otimes 1\otimes 1),$$

where $\varpi = (\varphi\otimes\psi)(\operatorname{Ad}W\tilde\nu)\in(K\otimes K)^*$. Therefore for any $a\in A$ we get

$$(\operatorname{Ad}\tilde\nu_{23}W^*_{23})(\iota\otimes\iota\otimes\varphi\otimes\iota\otimes\psi)(\iota\otimes((\lambda^\omega\boxtimes\lambda^{\bar\omega})\boxtimes(\lambda^\nu\boxtimes\lambda^{\bar\nu})))\delta(a) = \varpi_{35}(\eta^{\omega\nu}\delta(a)\otimes 1\otimes 1).$$

This shows that $\operatorname{Ad}(\tilde\nu_{23}W^*_{23})$ maps the algebra $(A_\omega)_\nu$ onto $A_{\omega\nu}\otimes 1$, which proves the first part of the proposition. Then the second part also follows, since the deformation of A by the trivial cocycle is equal to $\delta(A)$. □

2.3 K-theory

We say that two cocycles $\omega_0,\omega_1\in Z^2(G;\mathbb{T})$ are homotopic if there exists a $C([0,1];\mathbb{T})$-valued Borel 2-cocycle Ω on G such that $\omega_i=\Omega(\cdot,\cdot)(i)$ for $i=0,1$. Our goal is to show that under certain assumptions on G the deformed algebras A_{ω_0} and A_{ω_1} have isomorphic K-theory. For this we will use the following slight generalization of invariance under homotopy of cocycles of K-theory of reduced twisted group C*-algebras, proved in [20].

Theorem 2.3.1. *Assume G satisfies the Baum-Connes conjecture with coefficients. Then for any action α of G on a C*-algebra B and any two homotopic cocycles $\omega_0,\omega_1\in Z^2(G;\mathbb{T})$, for the corresponding reduced twisted crossed products we have $K_*(B\rtimes_{\alpha,\omega_0}G)\cong K_*(B\rtimes_{\alpha,\omega_1}G)$.*

The proof follows the same lines as that of [20, Theorem 1.9]. The starting point is the isomorphism

$$K \otimes (B \rtimes_{\alpha,\omega} G) \cong (K \otimes B) \rtimes_{\operatorname{Ad}\rho^{\bar\omega}\otimes\alpha} G, \quad x \mapsto \omega_{13}^* V_{13} x V_{13}^* \omega_{13},$$

which maps $\rho_g^{\bar\omega} \otimes 1 \otimes \rho_g^{\omega}$ into $1 \otimes 1 \otimes \rho_g$. This is a particular case of the Packer-Raeburn stabilization trick, see [49, Section 3]. Therefore instead of twisted crossed products we can consider $(K \otimes B) \rtimes_{\operatorname{Ad}\rho^{\bar\omega}\otimes\alpha} G$.

Now, given a homotopy Ω of cocycles, consider the action $\operatorname{Ad}\rho^{\bar\Omega}$ of G on $C[0,1]\otimes K$ defined, upon identifying $C[0,1] \otimes K$ with $C([0,1];K)$, by $(\operatorname{Ad}\rho_g^{\bar\Omega})(f)(t) = (\operatorname{Ad}\rho_g^{\bar\omega_t})(f(t))$, where $\omega_t = \Omega(\cdot,\cdot)(t)$.

Lemma 2.3.2 (cf. Proposition 1.5 in [20]). *For any compact subgroup $H \subset G$ and any $t \in [0,1]$, the restrictions of the actions $\operatorname{Ad}\rho^{\bar\Omega}$ and $\mathrm{id}\otimes\operatorname{Ad}\rho^{\bar\omega_t}$ to H are exterior equivalent.*

Note that this is easy to see for homotopies of the form $\omega_t = \omega_0 e^{itc}$ usually considered in applications, where c is an $\mathbb{R}$-valued Borel 2-cocycle. Indeed, by [40, Theorem 2.3] the second cohomology of a compact group with coefficients in $\mathbb{R}$ is trivial, so there exists a Borel function $b\colon H \to \mathbb{R}$ such that $c(h',h) = b(h') + b(h) - b(h'h)$. Extend b to a function on G as follows. Choose a Borel section $s\colon G/H \to G$ of the quotient map $G \to G/H$, $g \mapsto \dot g$, such that $s(\dot e) = e$. Then put

$$b(g) = b(s(\dot g)^{-1}g) - c(s(\dot g), s(\dot g)^{-1}g) + b(e).$$

A simple computation shows that $c(g,h) = b(g) + b(h) - b(gh)$ for all $g \in G$ and $h \in H$. Then the unitaries $u_h \in M(C[0,1]\otimes K)$ defined by $u_h(t) = e^{it(b-b(\cdot h))}$ form a 1-cocycle for the action $(\operatorname{Ad}\rho^{\bar\Omega})|_H$ such that $\operatorname{Ad}(u_h\rho_h^{\bar\Omega}) = \mathrm{id}\otimes\operatorname{Ad}\rho_h^{\bar\omega_0}$.

Proof of Theorem 2.3.1. For every $t \in [0,1]$ consider the evaluation map $\mathrm{ev}_t\colon C[0,1]\otimes K\otimes B \to K\otimes B$. It is G-equivariant with respect to the action $\operatorname{Ad}\rho^{\bar\Omega}\otimes\alpha$ of G on $C[0,1]\otimes K\otimes B$ and the action $\operatorname{Ad}\rho^{\bar\omega_t}\otimes\alpha$ of G on $K\otimes B$. We claim that it induces an isomorphism

$$(\mathrm{ev}_t \rtimes G)_*\colon K_*((C[0,1]\otimes K\otimes B) \rtimes_{\operatorname{Ad}\rho^{\bar\Omega}\otimes\alpha} G) \to K_*((K\otimes B)\rtimes_{\operatorname{Ad}\rho^{\bar\omega_t}\otimes\alpha} G).$$

By [20, Proposition 1.6] in order to show this it suffices to check that for every compact subgroup H of G the map ev_t induces an isomorphism

$$(\mathrm{ev}_t \rtimes H)_*\colon K_*((C[0,1]\otimes K\otimes B) \rtimes_{\operatorname{Ad}\rho^{\bar\Omega}\otimes\alpha} H) \to K_*((K\otimes B)\rtimes_{\operatorname{Ad}\rho^{\bar\omega_t}\otimes\alpha} H).$$

By Lemma 2.3.2 the action $\operatorname{Ad}\rho^{\bar\Omega}\otimes\alpha$ of H on $C[0,1]\otimes K\otimes B$ is exterior equivalent to the action $\mathrm{id}\otimes\operatorname{Ad}\rho^{\bar\omega_t}\otimes\alpha$, so that

$$(C[0,1]\otimes K\otimes B) \rtimes_{\operatorname{Ad}\rho^{\bar\Omega}\otimes\alpha} H \cong C[0,1]\otimes((K\otimes B)\rtimes_{\operatorname{Ad}\rho^{\bar\omega_t}\otimes\alpha} H).$$

If the cocycle $U = \{u_h\}_{h\in H}$ defining the exterior equivalence is chosen such that $u_h(t) = 1$ for all $h \in H$, then the corresponding homomorphism

$$C[0,1]\otimes((K\otimes B)\rtimes_{\operatorname{Ad}\rho^{\bar\omega_t}\otimes\alpha} H) \to (K\otimes B)\rtimes_{\operatorname{Ad}\rho^{\bar\omega_t}\otimes\alpha} H$$

is simply the evaluation at t. Obviously, it defines an isomorphism in K-theory. □

Combining Theorems 2.1.4 and 2.3.1 we get the following result that generalizes several earlier results in the literature [53, 62].

Corollary 2.3.3. *Assume G satisfies the Baum-Connes conjecture with coefficients. Then for any coaction δ of G on a C^*-algebra A and any two homotopic cocycles $\omega_0, \omega_1 \in Z^2(G;\mathbb{T})$, we have an isomorphism $K_*(A_{\omega_0}) \cong K_*(A_{\omega_1})$.*

We finish by noting that for some groups it is possible to prove a stronger result. For example, generalizing Rieffel's result for $\mathbb{R}^d$ [53] we have the following.

Proposition 2.3.4. *If G is a simply connected solvable Lie group, then for any coaction δ of G on a C^*-algebra A and any cocycle $\omega \in Z^2(G;\mathbb{T})$ we have $K_*(A_\omega) \cong K_*(A)$.*

Proof. By the stabilization trick and Connes' Thom isomorphism we have $K_i(A \rtimes_\delta \widehat{G} \rtimes_{\hat{\delta},\omega} G) \cong K_{i+\dim G}(A \rtimes_\delta \widehat{G}) \cong K_i(A \rtimes_\delta \widehat{G} \rtimes_{\hat{\delta}} G) \cong K_i(A)$. $\square$

2.4 Deformation of Cuntz-Pimsner algebras

As there are well-defined notions of actions/coactions on C*-correspondences with resulting crossed product C*-correspondences, it should be a natural question to ask how deformation would relate to this picture. We shall consider a coaction of a discrete group G with a 2-cocycle ω. The general idea is roughly: starting with a C*-correspondence (X, A) equipped with a coaction, construct a new C*-correspondence (X_ω, A_ω) - the deformed C*-correspondence. Concerning the Cuntz-Pimsner algebra $\mathcal{O}_X$ of the original C*-correspondence (X, A), one gets a naturally induced action/coaction from which one may construct the deformation $(\mathcal{O}_X)_\omega$. On the other hand we may consider the Cuntz-Pimsner algebra $\mathcal{O}_{X_\omega}$ of the deformed C*-correspondence (X_ω, A_ω). An interesting question then is to relate $(\mathcal{O}_X)_\omega$ and $\mathcal{O}_{X_\omega}$ as illustrated in the diagram:

$$\begin{tikzcd}
(X,A) \arrow[r, "\text{Cuntz-Pimsner}"] \arrow[d, "\text{deformation}"'] & \mathcal{O}_X \arrow[r, "\text{deformation}"] & (\mathcal{O}_X)_\omega \arrow[d, "\text{isomorphic ?}"] \\
(X_\omega, A_\omega) \arrow[rr, "\text{Cuntz-Pimsner}"'] & & \mathcal{O}_{X_\omega}
\end{tikzcd}$$

We aim to show that

$$(\mathcal{O}_X)_\omega \cong \mathcal{O}_{X_\omega}.$$

Coaction and deformation

Let (X, A) be a C*-correspondence endowed with a coaction (σ, δ) of a discrete group G, meaning (see section 1.5)

(i) $\delta : A \longrightarrow M(A \otimes C_r^*(G))$ is a coaction of G on A, i.e. a non-degenerate injective *-homomorphism satisfying $(\delta \otimes 1)\delta = (1 \otimes \delta_G)\delta$ and $\overline{span}\{\delta(A)(1 \otimes C_r^*(G))\} = A \otimes C_r^*(G)$,

(ii) $\sigma : X \longrightarrow M(X \otimes C_r^*(G))$ is a coaction of G on X, i.e. a linear map satisfying $(\sigma \otimes 1)\sigma = (1 \otimes \delta_G)\sigma$ and $\overline{span}\{\sigma(X)(1 \otimes C_r^*(G))\} = X \otimes C_r^*(G)$,

(iii) $\sigma(a \cdot x) = \delta(a) \cdot \sigma(x)$, for $a \in A$, $x \in X$,

(iv) $\sigma(x \cdot b) = \sigma(x) \cdot \delta(a)$ for $b \in A$, $x \in X$,

(v) $\delta(\langle x, y \rangle) = \langle \sigma(x), \sigma(y) \rangle$, for $x, y \in X$.

For each $g \in G$ the g-component of A is the subspace

$$A_g = \{a \in A : \delta(a) = a \otimes \lambda_g\}.$$

Similarly the g-component of X is the subspace

$$X_g = \{x \in X : \sigma(x) = x \otimes \lambda_g\}.$$

Let $\omega \in Z^2(G, \mathbb{T})$ be a 2-cocycle. In the current setting (see Example 2.1.2) the deformed algebra A_ω can be identified as

$$A_\omega = \overline{span}\{a \otimes \lambda_g^\omega : g \in G, a \in A_g\} \subset A \otimes C^*(G, \omega).$$

According to this description of the deformed algebra, we give the following definition of the twisted module

$$X_\omega = \overline{span}\{x \otimes \lambda_g^\omega : g \in G, x \in X_g\} \subset X \otimes C^*(G, \omega),$$

where $X \otimes C^*(G, \omega)$ is the obvious external tensor product C*-module.

Lemma 2.4.1. (X_ω, A_ω) *is a C*-correspondence.*

Proof. First we show that the right A_ω-action on X_ω is well-defined: Let $a \in A_g$ and $x \in X_h$, so that $a \otimes \lambda_g^\omega \in A_\omega$ and $x \otimes \lambda_h^\omega \in X_\omega$. As $\sigma(xa) = \sigma(x)\delta(a) = (x \otimes \lambda_m)(a \otimes \lambda_g) = xa \otimes \lambda_{hg}$ it follows that $xa \in X_{hg}$. Thus it is clear that $(x \otimes \lambda_h^\omega)(a \otimes \lambda_g^\omega) = xa \otimes \omega(h, g)\lambda_{hg}^\omega$ belongs to X_ω. It follows from this that X_ω is a right A_ω-module. The left A_ω-action on X_ω is checked in a similar way.

Next, for the A_ω-valued inner product on X_ω: Let $x \in X_g$ and $y \in X_h$, so that $x \otimes \lambda_g^\omega \in X_\omega$ and $y \otimes \lambda_h^\omega \in X_h$. Then $\delta(\langle x, y \rangle) = \langle \sigma(x), \sigma(y) \rangle = \langle x \otimes \lambda_g, y \otimes \lambda_h \rangle = \langle x, y \rangle \otimes \lambda_g^* \lambda_h = \langle x, y \rangle \otimes \lambda_{g^{-1}h}$ shows that $\langle x, y \rangle \in X_{g^{-1}h}$, and it follows that $\langle x \otimes \lambda_g^\omega, y \otimes \lambda_h^\omega \rangle = \langle x, y \rangle \otimes \lambda_g^{\omega *} \lambda_h^\omega \in A_\omega$. $\square$

Recall that (k_X, k_A) is the universal covariant representation, $k_X : X \longrightarrow \mathcal{O}_X$ and $k_A : A \longrightarrow \mathcal{O}_X$, and $\mathcal{O}_X = C^*(k_X, k_A)$. The coaction (σ, δ) on (X, A) induces naturally a coaction $\zeta : \mathcal{O}_X \longrightarrow M(\mathcal{O}_X \otimes C_r^*(G))$ by

$$\zeta(k_X(x)) = (k_X \otimes 1)\sigma(x), \quad \zeta(k_A(a)) = (k_A \otimes 1)\delta(a),$$

for $x \in X$, $a \in A$. We get components

$$(\mathcal{O}_X)_g = \{h \in \mathcal{O}_X : \zeta(h) = h \otimes \lambda_g\}, \quad g \in G.$$

If $x \in X_g$ then $\zeta(k_X(x)) = k_X(x) \otimes \lambda_g$, which means that $k_X(X_g) \subset (\mathcal{O}_X)_g$. Likewise for $a \in A_g$, $\zeta(k_A(a)) = k_A(a) \otimes \lambda_g$ so that $k_A(A_g) \subset (\mathcal{O}_X)_g$.

The cocycle deformation $(\mathcal{O}_X)_\omega$ with respect to the coaction ζ and 2-cocycle ω, can be identified with

$$(\mathcal{O}_X)_\omega = \overline{span}\{w \otimes \lambda_g^\omega : g \in G, w \in (\mathcal{O}_X)_g\}.$$

On the other hand we can consider the Cuntz-Pimsner algebra $\mathcal{O}_{X_\omega}$ of our deformed C*-correspondence (X_ω, A_ω). We denote the universal covariant representation $(k_{X_\omega}, k_{A_\omega})$ so that $\mathcal{O}_{X_\omega} = C^*(k_{X_\omega}, k_{A_\omega})$.

Theorem 2.4.2. *There is an isomorphism $\mathcal{O}_{X_\omega} \cong (\mathcal{O}_X)_\omega$ mapping*

$$k_{X_\omega}(x \otimes \lambda_g^\omega) \mapsto k_X(x) \otimes \lambda_g^\omega, \qquad k_{A_\omega}(a \otimes \lambda_h^\omega) \mapsto k_A(a) \otimes \lambda_h^\omega,$$

for $x \in X_g$ and $a \in A_h$.

Proof. We define a covariant representation (π, t) of (X_ω, A_ω) on $(\mathcal{O}_X)_\omega$. Namely, let $a \in A_h$ and $x \in X_g$, then $\pi : A_\omega \longrightarrow (\mathcal{O}_X)_\omega$, $\pi(a \otimes \lambda_h^\omega) = k_A(a) \otimes \lambda_h^\omega$, and $t : X_\omega \longrightarrow (\mathcal{O}_X)_\omega$, $t(x \otimes \lambda_g^\omega) = k_X(x) \otimes \lambda_g^\omega$. That is, $\pi = k_A \otimes 1$ and $t = k_X \otimes 1$. It should be clear from construction that $C^*(\pi, t) = (\mathcal{O}_X)_\omega$. It is also clear that (π, t) is an injective covariant representation, and moreover it admits a gauge action: clearly from the standard gauge action γ for the universal covariant representation (k_X, k_A) we get that $\gamma \otimes 1$ serves as a gauge action for (π, t). It then follows from the gauge-invariant uniqueness theorem, [32, Theorem 6.4] that the natural surjection $\mathcal{O}_{X_\omega} \longrightarrow (\mathcal{O}_X)_\omega$ is an isomorphism. This is our claimed map. $\square$

Chapter 3

Rieffel deformation and KK-fibrations

In the present chapter we specialize to abelian groups and continuous cocycles, thereby considering the setup of Kasprzak. We will prove a K-theory isomorphism using different methods than those already considered, namely by using $C_0(X)$-algebras and (R)KK-fibrations.

3.1 Deformation by actions of abelian groups

We first recall the Landstad-Quigg-Vaes theorem. Assume we are given an action α of G on a C*-algebra B and a nondegenerate homomorphism $\pi : C_0(G) \longrightarrow M(B)$ such that $\alpha(\pi(f)) = (\pi \otimes \iota)\Delta(f)$. Put $X = (\pi \otimes \iota)(W)$ and consider the homomorphism

$$\eta : B \longrightarrow M(B \otimes K), \quad \eta(x) = X\alpha(x)X^*.$$

Then the closed linear span $A \subseteq M(B)$ of elements of the form $(\iota \otimes \varphi)\eta(x)$ with $x \in B$ and $\varphi \in K^*$ is a C*-algebra, the formula $\delta(a) = X(a \otimes 1)X^*$ defines a coaction of G on A, and η becomes an isomorphism $B \cong A \rtimes_\delta \hat{G}$ that intertwines α with $\hat{\delta}$. A useful remark is that $A \subseteq M(B)^\alpha$. It also follows that the inverse of $\eta : B \longrightarrow A \rtimes_\delta \hat{G}$ is given by $\delta(a)(1 \otimes f) \mapsto a\pi(f)$.

Let G be a locally compact abelian group. Recall the Fourier transform is given by conjugation by the unitary operator $U : L^2(G) \longrightarrow L^2(\hat{G})$, $(U\xi)(g) = \int_G \xi(g)\bar{\chi}(g)\, dg$, where we assume the Haar measure on $\hat{G}$ to be normalized. Let m_χ denote the operator of multiplication by the function χ on $L^2(G)$. Similarly, let m_g denote the operator of multiplication by the function $\chi \mapsto \chi(g)$ on $L^2(\hat{G})$. Denote by $\widehat{W}$ the multiplicative unitary of $\hat{G}$. Then we have

$$Um_\chi = \lambda_\chi U, \quad U\rho_g = m_g U, \quad (U \otimes U)V = \widehat{W}(U \otimes U).$$

Assume we have an action α of G on the C*-algebra A. It defines a coaction δ of $\hat{G}$ by

$$\delta(a) = (1 \otimes U)\alpha(a)(1 \otimes U^*), \quad a \in A.$$

We will translate our formulas for deformation for δ and $\hat{G}$ in terms of α and G. The crossed product $A \rtimes_\alpha G$ is represented in $M(A \otimes K(L^2(G)))$ by $a \mapsto \alpha(a)$,

$\lambda_g \mapsto 1 \otimes \rho_g$. Then we have

$$Ad(1 \otimes U^*)(A \rtimes_\alpha G) = A \rtimes_\delta \widehat{G} = \overline{\delta(A)(1 \otimes C_0(\widehat{G}))}.$$

Let ω be a 2-cocycle on $\widehat{G}$. In order to simplify matters, and since it will be sufficient for our applications, assume that ω is a skew-symmetric bi-character so that ω is multiplicative in each variable and $\omega(\chi, \chi) = 1$. Then we can define a continuous homomorphism $r : \widehat{G} \longrightarrow G$ such that $\omega(\chi, \chi') = \chi(r(\chi'))$.

The twisted dual action $\widehat{\delta}^\omega = \widetilde{\omega}_{23}\widehat{\delta}(\cdot)\widetilde{\omega}_{23}^*$ on $A \rtimes_\delta \widehat{G}$ defines an action $\widehat{\alpha}^\omega$ on $A \rtimes_\alpha G$ by

$$\widehat{\alpha}^\omega(a) = Ad(1 \otimes U^* \otimes 1)\widehat{\delta}^\omega(Ad(1 \otimes U)(x)).$$

Without ω this would have been the usual dual action

$$\widehat{\alpha}_\chi(\alpha(a)) = \alpha(a), \quad \widehat{\alpha}_\chi(1 \otimes \rho_g) = \chi(g)(1 \otimes \rho_g).$$

With ω we get

$$\widehat{\alpha}_\chi^\omega(x) = Ad((1 \otimes U^*)(1 \otimes \widetilde{\omega}(\cdot, \chi))(1 \otimes U))\widehat{\alpha}_\chi(x).$$

As $\widetilde{\omega} = \bar{\omega}$, $\bar{\omega}(\cdot, \chi) = m_{r(\chi)}^*$ and $U^* m_{r(\chi)}^* U = \rho_{-r(\chi)}$, this implies

$$\widehat{\alpha}_\chi^\omega(x) = Ad(1 \otimes \rho_{-r(\chi)})\widehat{\alpha}_\chi(x). \tag{3.1.1}$$

Therefore

$$\widehat{\alpha}_\chi^\omega(\alpha(a)) = \alpha(\alpha_{-r(\chi)}), \quad \widehat{\alpha}_\chi^\omega(1 \otimes \rho_g) = \chi(g)(1 \otimes \rho_g).$$

Turning to $\eta^\omega = \widehat{W}_{23}\widehat{\delta}^\omega(\cdot)\widehat{W}_{23}^*$, this is a map

$$A \rtimes_\delta \widehat{G} \longrightarrow M(A \otimes K(L^2(\widehat{G})) \otimes K(L^2(\widehat{G}))).$$

It defines a map

$$A \rtimes_\alpha G \longrightarrow M(A \otimes K(L^2(G)) \otimes K(L^2(G))), x \mapsto Ad(1 \otimes U^* \otimes U^*)\eta^\omega(Ad(1 \otimes U)(x)),$$

which we also denote η^ω. Since $(U^* \otimes U^*)\widehat{W} = V(U^* \otimes U^*)$, we get

$$\eta^\omega(x) = (1 \otimes V)(1 \otimes 1 \otimes U^*)\widehat{\alpha}^\omega(x)(1 \otimes 1 \otimes U)(1 \otimes V^*). \tag{3.1.2}$$

Taking slices we get our deformed algebra $A_\omega \subset M(A \rtimes_\alpha G)$. The new coaction $\delta^\omega = \widehat{W}_{23}(\cdot \otimes 1)\widehat{W}_{23}^*$ of $\widehat{G}$. It defines an action α^ω of G on $A_\omega \subset M(A \rtimes_\alpha G)$ by

$$\alpha^\omega(x) = Ad(1 \otimes U^* \otimes U^*)\delta^\omega(Ad(1 \otimes U)(x)) = (1 \otimes V)(x \otimes 1)(1 \otimes V^*).$$

In other words

$$\alpha_g^\omega(x) = Ad(1 \otimes \rho_g)(x). \tag{3.1.3}$$

This is how Landstad's theory defines the action.

Now assume in addition that G is *almost periodic*, that is, A is generated by the spectral subspaces $A_\chi = \{a \in A : \alpha_g(a) = \chi(g)a\}$. When we write a_χ, we mean that a_χ is an element of A_χ.

Lemma 3.1.1. *The C^*-algebra $A_\omega \subset M(A \rtimes_\alpha G)$ is generated by elements of the form $a_\chi \lambda_{-r(\chi)}$.*

Proof. Using the standard representation of the crossed product, the lemma could rather be formulated as the statement that A_ω is generated by elements $\alpha(a_\chi)(1\otimes\rho_{-r(\chi)})$. By (3.1.1) we have

$$\hat{\alpha}^\omega_{\chi'}(\alpha(a_\chi)) = \alpha(\alpha_{-r(\chi)}(a_\chi)) = \chi(-r(\chi'))\alpha(a_\chi) = \chi'(r(\chi))\alpha(a_\chi),$$

that is, $\hat{\alpha}^\omega(\alpha(a_\chi)) = \alpha(a_\chi)\otimes m_{r(\chi)} = a_\chi\otimes m_\chi\otimes m_{r(\chi)}$. It follows from (3.1.2) that

$$\eta^\omega(\alpha(a_\chi)) = (1\otimes V)(a_\chi\otimes m_\chi\otimes\rho_{r(\chi)})(1\otimes V^*).$$

Now, $V(m_\chi\otimes 1)V^* = m_\chi\otimes m_\chi$ and $V(1\otimes\rho_{r(\chi)})V^* = \rho_{-r(\chi)}\otimes\rho_{r(\chi)}$. Therefore

$$\eta^\omega(\alpha(a_\chi)) = a_\chi\otimes m_\chi\rho_{-r(\chi)}\otimes m_\chi\rho_{r(\chi)} = (\alpha(a_\chi)\otimes 1)(1\otimes\rho_{-r(\chi)}\otimes m_\chi\rho_{r(\chi)}).$$

Taking slices, we get the result. □

Note that the proof of the lemma basically repeats Example 2.1.2, and using that example we quickly arrive at the same conclusion. Indeed, as follows from Example 2.1.2, A_ω is generated by the elements $a_\chi\otimes U^*\lambda^\omega_\chi U$. But we have $\lambda^\omega_\chi = \tilde{\omega}(\chi^{-1},\cdot)\lambda_\chi = m_{-r(\chi)}\lambda_\chi$. Hence A_ω is generated by the elements

$$a_\chi\otimes\rho_{-r(\chi)}m_\chi = (1\otimes\rho_{-r(\chi)})\alpha(a_\chi) = \alpha(a_\chi)(1\otimes\rho_{-r(\chi)}).$$

As discussed in the introductory remark on the Landstad-Quigg-Vaes theorem above, the isomorphism $A_\omega\rtimes_{\alpha^\omega}G\cong A\rtimes_\alpha G$ is defined by using the embedding $A_\omega\hookrightarrow M(A\rtimes_\alpha G)$ and the identity map on $C^*(G)$. We can rephrase what we have above as follows. Denote by $\mathcal{A}$ the *-algebra spanned by the spectral subspaces A_χ. We can define a new product $\times_\omega$ on $\mathcal{A}$ by

$$a_\chi\times_\omega b_{\chi'} = \omega(\chi,\chi')a_\chi b_{\chi'}.$$

Denote by $\mathcal{A}_\omega$ the algebra $\mathcal{A}$ with this new product $\times_\omega$. We can embed $\mathcal{A}_\omega$ into $M(A\rtimes_\alpha G)$ by $a_\chi\mapsto a_\chi\lambda_{-r(\chi)}$, or in other words $a_\chi\mapsto\alpha(a_\chi)(1\otimes\rho_{-r(\chi)})$. This gives a C*-norm on $\mathcal{A}_\omega$ and we define A_ω as the completion of $\mathcal{A}_\omega$. The same completion (in fact, modulo conjugating by $1\otimes U$, the same embedding) is obtained by using the map $\mathcal{A}_\omega\hookrightarrow M(A\otimes C^*(\hat{G},\omega))$, $a_\chi\mapsto a_\chi\otimes\lambda^\omega_\chi$. The action $Ad\,\lambda$ on $A\rtimes_\alpha G$ defines an action α^ω of G on A_ω. On $\mathcal{A}_\omega=\mathcal{A}$ this is just the original action α. Then the embedding $A_\omega\hookrightarrow M(A\rtimes_\alpha G)$, together with the identity map $\lambda_g\mapsto\lambda_g$, defines the isomorphism $A_\omega\rtimes_{\alpha^\omega}G\cong A\rtimes_\alpha G$.

3.2 Rieffel's deformation

It was stated by Kasprzak [31] that for $G=\mathbb{R}^d$ his approach to deformation, which our construction of chapter 2 extends, is equivalent to that of Rieffel [52], but no proof of this was given. A sketch of a possible proof was then proposed by Hannabuss and Mathai [24], but in our opinion their arguments raise more questions than provide answers. The goal of this section is to give a different rigorous proof using completely positive maps constructed by Kaschek, Neumaier and Waldmann [34].

We will use the conventions in [34] that are slightly different from those of Rieffel. Assume V is a $2n$-dimensional Euclidean space with scalar product $\langle\cdot,\cdot\rangle$, and J is a complex structure on V, so J is an orthogonal transformation and $J^2 = -1$. Fix a deformation parameter $h > 0$.

Assume we are given an action α of V on a C*-algebra A. Denote by A_∞ the subalgebra of smooth vectors for this action. It is a Fréchet algebra equipped with differential norms $\|\cdot\|_k$, $k \geqslant 1$. Rieffel defines a new product $*_h$ on A_∞ by

$$a *_h b = \frac{1}{(\pi h)^{2n}} \int_{V\times V} \alpha_x(a)\alpha_y(b) e^{-\frac{2i}{h}\langle x, Jy\rangle} dx\, dy, \tag{3.2.1}$$

where the integral is understood as an oscillatory integral. Denote by A_x the spectral subspace of A_∞ corresponding to $x \in V$, so A_x consists of elements $a \in A$ such that $\alpha_y(a) = e^{i\langle x,y\rangle} a$ for all $y \in V$. Then for $a \in A_x$ and $b \in A_y$ we have

$$a *_h b = e^{\frac{ih}{2}\langle x, Jy\rangle} ab. \tag{3.2.2}$$

Therefore the cocycle of deformation is $\omega(x, y) = e^{\frac{ih}{2}\langle x,Jy\rangle}$. The Rieffel deformation of A is a certain C*-algebraic completion of A_∞ equipped with the product $*_h$ and with the involution inherited from A, see [52] for details. We denote it by $\tilde{A}_\omega$.

As shown in the previous section, conjugating the action α by the Fourier transform we get a coaction δ of on A. In this section we define the Fourier transform $U : L^2(V) \to L^2(V)$ by

$$(Uf)(x) = \frac{1}{(2\pi)^n} \int_V f(y) e^{-i\langle x,y\rangle} dy.$$

Then $\mathrm{Ad}\, U$ defines an isomorphism of $C_0(V)$ onto $C^*_r(V)$, and by letting $\delta = \mathrm{Ad}(1 \otimes U)\alpha$ we get a coaction of V on A. Note that $a \in A$ lies in the spectral subspace A_x if and only if $\delta(a) = a \otimes \lambda_x$, in agreement with our notation in chapter 1. We can then consider the ω-deformation A_ω of A. Our aim is to construct an isomorphism between A_ω and $\tilde{A}_\omega$.

Following [34] define a map $\Phi\colon A \to A$ by

$$\Phi(a) = \frac{1}{(\pi h)^n} \int_V e^{-\frac{1}{h}\|x\|^2} \alpha_x(a) dx.$$

We have

$$\Phi(a) = e^{-\frac{h}{4}\|x\|^2} a \quad \text{for} \quad a \in A_x. \tag{3.2.3}$$

The image of Φ is contained in A_∞. So we can consider Φ as a map $\tilde{T}\colon A \to \tilde{A}_\omega$. Identifying A with Rieffel's deformation of $\tilde{A}_\omega$ corresponding to the complex structure $-J$, we also get a similarly defined map $\tilde{S}\colon \tilde{A}_\omega \to A$, so the restriction of $\tilde{S}$ to A_∞ coincides with Φ. Since Φ considered as a map $(A, \|\cdot\|) \to (A_\infty, \|\cdot\|_k)$ is bounded for any k, the map $\tilde{T}\colon A \to \tilde{A}_\omega$ is bounded by standard estimates for the operator norm on $\tilde{A}_\omega$, see [52, Proposition 4.10]. By symmetry $\tilde{S}$ is also bounded. The main result in [34] states that the maps $\tilde{T}$ and $\tilde{S}$ are completely positive. We will reprove this a bit later.

We want to define analogues of the maps $\tilde{T}$ and $\tilde{S}$ for A_ω. For this define a unit vector $\xi_0 \in L^2(V)$ by

$$\xi_0(x) = \left(\frac{h}{2\pi}\right)^{n/2} e^{-\frac{h}{4}\|x\|^2}.$$

Consider the normal state $\varphi_0 = (\cdot\, \xi_0, \xi_0)$ on $B(L^2(V))$. We have

$$\varphi_0(\lambda_x^\omega) = \varphi_0(\lambda_x^{\bar{\omega}}) = e^{-\frac{h}{4}\|x\|^2}.$$

Define $T\colon A \to A_\omega$ and $S\colon A_\omega \to A$ by

$$T(a) = (\iota \otimes \iota \otimes \varphi_0)\eta^\omega \delta(a), \quad S(b) = (\iota \otimes \varphi_0)(b).$$

Clearly, these are completely positive maps. Using that $\eta^\omega\delta(a) = a \otimes \lambda_x^\omega \otimes \lambda_x^{\bar{\omega}}$ for $a \in A_x$, we get

$$T(a) = e^{-\frac{h}{4}\|x\|^2} a \otimes \lambda_x^\omega \quad \text{and} \quad S(a \otimes \lambda_x^\omega) = e^{-\frac{h}{4}\|x\|^2} a \quad \text{for} \quad a \in A_x. \tag{3.2.4}$$

Lemma 3.2.1. *For any $n \geqslant 1$ and $a_1, \dots, a_n \in A$ we have*

$$\tilde{S}(\tilde{T}(a_1) \dots \tilde{T}(a_n)) = S(T(a_1) \dots T(a_n)).$$

Proof. If for every j the element a_j lies in a spectral subspace A_{x_j}, then the identity in the formulation follows immediately from (3.2.3) and (3.2.4). We will show that this is enough to conclude that it holds for arbitrary elements.

We claim that there exists a von Neumann algebra M containing A such that the action α of V on A extends to a continuous (in the von Neumann algebraic sense) action of V on M and such that M is generated as a von Neumann algebra by the spectral subspaces of this action. Indeed, first represent the crossed product $A \rtimes_\alpha V$ faithfully on some Hilbert space H and consider the von Neumann algebra $N \subset B(H)$ generated by A. The action α of V on A extends to an action β of V on N. Consider the double crossed product $M = N \rtimes_\beta V \rtimes_{\hat{\beta}} \hat{V}$ in the von Neumann algebraic sense. By the Takesaki-Takai duality we have $(M, \hat{\hat{\beta}}) \cong (N \bar{\otimes} B(L^2(V)), \beta \otimes \operatorname{Ad}\rho)$. This gives us an equivariant embedding of $A \subset N$ into M equipped with the action $\hat{\hat{\beta}}$. It is also clear that M is generated by the spectral subspaces of the action, so our claim is proved.

We continue to denote by α the action of V on M. Denote by $\mathcal{M} \subset M$ the set of elements $a \in M$ such that the map $x \mapsto \alpha_x(a)$ is norm-continuous. This is an ultrastrongly operator dense C*-subalgebra of M. We continue to denote by $T, S, \tilde{T}, \tilde{S}$ the maps defined for the C*-algebra $\mathcal{M}$ in place of A. The maps T and S have obvious extensions to normal maps between the von Neumann algebras generated by $\mathcal{M}$ and $\mathcal{M}_\omega$. On the other hand, the map Φ,

$$\Phi(a) = \frac{1}{(\pi h)^n} \int_V e^{-\frac{1}{h}\|x\|^2} \alpha_x(a) dx,$$

is still well-defined on M, but now the integral should be taken with respect to the ultrastrong operator topology. The image of M under Φ is contained in $\mathcal{M}_\infty$. It therefore makes sense to ask whether the identity

$$\Phi(\Phi(a_1) *_h \dots *_h \Phi(a_n)) = S(T(a_1) \dots T(a_n))$$

holds for all $a_1, \dots, a_n \in M$, which would imply the assertion of the lemma. Since this identity holds for $a_1, \dots, a_n$ lying in spectral subspaces of M, it suffices to show that both sides of the identity are normal maps in every variable a_j running through the unit ball M_1 of M. This is clearly the case for the right hand side. In order to prove the same for the left hand side it suffices to show that for any $b, c \in \mathcal{M}_\infty$ the map

$$M_1 \to M, \quad a \mapsto \Phi(b *_h \Phi(a) *_h c),$$

is continuous in the ultrastrong operator topology.

Using basic estimates for oscillatory integrals, see [52, Chapter 1], and the fact that the map Φ is bounded as a map $(M, \|\cdot\|) \to (\mathcal{M}_\infty, \|\cdot\|_k)$ for every k, it is easy to check that $\Phi(b *_h \Phi(a) *_h c)$ can be approximated in norm uniformly in $a \in M_1$ by integrals of the form

$$\int_{V^3} \psi(x, y, z)\alpha_x(b)\alpha_y(a)\alpha_z(c)dx\,dy\,dz,$$

where ψ is a smooth compactly supported function and the integral is taken with respect to the ultrastrong operator topology. Since such integrals are clearly continuous in $a \in M_1$ in this topology, this finishes the proof of the lemma. □

We will need the above lemma only for $n = 1, 2, 3$.

Lemma 3.2.2. *The maps $\tilde{T}$ and $\tilde{S}$ are completely positive, and all four maps $T, S, \tilde{T}, \tilde{S}$ are faithful and their images are dense.*

Proof. We begin by proving that the images of $\tilde{T}$ and $\tilde{S}$ are dense. It suffices to consider $\tilde{S}$, and then it is enough to show that the image of Φ is dense. Note that

$$\Phi(\alpha_y(a)) = \frac{1}{(\pi h)^n} \int_V e^{-\frac{1}{h}\|x-y\|^2} \alpha_x(a)dx.$$

It is well-known, and is easy to check using e.g. Wiener's Tauberian theorem, that the translations of the function $e^{-\frac{1}{h}\|x\|^2}$ span a dense subspace of $L^1(V)$. It follows that the closure of the image of Φ contains all elements of the form $\int_V f(x)\alpha_x(a)dx$ with $f \in L^1(V)$. Hence this closure coincides with A.

It is clear that $\tilde{T}(a) \neq 0$ for any nonzero $a \geqslant 0$, and by symmetry $\tilde{S}$ has the same property. We will next show that $\tilde{T}$ and $\tilde{S}$ are completely positive. It is enough to consider $\tilde{S}$. Since by Lemma 3.2.1 we have

$$\tilde{S}(\tilde{T}(a)^*\tilde{T}(a)) = S(T(a)^*T(a)) \geqslant 0,$$

and the image of $\tilde{T}$ is dense, we see that $\tilde{S}$ is positive. Passing to deformations of matrix algebras over A we conclude that $\tilde{S}$ is completely positive. This finishes the proof of the lemma for $\tilde{T}$ and $\tilde{S}$.

Turning to T and S, by Lemma 3.2.1 we have $ST = \tilde{S}\tilde{T} = \Phi^2$. Since the map Φ is faithful and its image is dense, it follows that the map T is faithful and the image of S is dense. Consider the maps $T'\colon A_\omega \to (A_\omega)_{\bar{\omega}}$ and $S'\colon (A_\omega)_{\bar{\omega}} \to A_\omega$ defined by $(A_\omega, \delta^\omega)$ in the same way as T and S were defined by (A, δ). Then T' is faithful and the image of S' is dense. By Proposition 2.2.4 the map $\eta^\omega\delta$ defines an isomorphism $A \cong (A_\omega)_{\bar{\omega}}$. By definition of T and S' we immediately get $T = S'\eta^\omega\delta$. Hence the image of T is dense. We also have $\eta^\omega\delta S = T'$. Indeed, a simple computation similar to the ones used in the proof of Proposition 2.2.4 shows that for $b = (\iota \otimes \iota \otimes \varphi)\eta^\omega\delta(a) \in A_\omega$ we have

$$\eta^\omega\delta S(b) = (\iota \otimes \iota \otimes \iota \otimes \varphi_0 \otimes \varphi)(\iota \otimes ((\lambda^\omega \boxtimes \lambda^{\bar{\omega}}) \boxtimes (\lambda^\omega \boxtimes \lambda^{\bar{\omega}})))\delta(a)$$

and

$$T'(b) = (\iota \otimes \iota \otimes \varphi \otimes \iota \otimes \varphi_0)(\iota \otimes ((\lambda^{\omega} \boxtimes \lambda^{\bar{\omega}}) \boxtimes (\lambda^{\bar{\omega}} \boxtimes \lambda^{\omega})))\delta(a).$$

Alternatively, the identity $\eta^{\omega}\delta S = T'$ is immediate on elements of the form $a \otimes \lambda_x$, where $a \in A_x$, hence it holds on arbitrary elements by an argument similar to the one used in the proof of Lemma 3.2.1. It follows that the map S is faithful. □

Note that despite the fact that S is faithful, the state φ_0 is very far from being faithful on the von Neumann algebra generated by $C^*_r(V, \omega)$. This von Neumann algebra is a factor of type I_∞ and φ_0 is a normal pure state on it. Indeed, the C*-algebra generated by the operators λ^{ω}_x is the algebra of canonical commutation relations for the space V equipped with the Hermitian scalar product $h\langle x, y\rangle + ih\langle x, Jy\rangle$, and φ_0 is nothing other than the vacuum state on it.

Theorem 3.2.3. *There is a unique isomorphism $A_\omega \cong \tilde{A}_\omega$ that maps $T(a)$ into $\tilde{T}(a)$ for every $a \in A$.*

Proof. Assume first that there exists a faithful state ψ on A. Consider the positive linear functionals $\psi_\omega = \psi S$ and $\tilde{\psi}_\omega = \psi\tilde{S}$ on A_ω and $\tilde{A}_\omega$. Since the positive maps S and $\tilde{S}$ are faithful, these functionals are faithful. Consider the faithful GNS-representation of A_ω on H with cyclic vector ξ defining ψ_ω, and the faithful GNS-representation of $\tilde{A}_\omega$ on $\tilde{H}$ with cyclic vector $\tilde{\xi}$ defining $\tilde{\psi}_\omega$. By Lemma 3.2.1 for $n = 2$ we have

$$(T(a)\xi, T(b)\xi) = (\tilde{T}(a)\tilde{\xi}, \tilde{T}(b)\tilde{\xi}).$$

Since the images of T and $\tilde{T}$ are dense, it follows that there exists a unitary operator $U\colon H \to \tilde{H}$ such that $UT(a)\xi = \tilde{T}(a)\tilde{\xi}$. By Lemma 3.2.1 for $n = 3$ we have

$$(T(a)T(b)\xi, T(c)\xi) = (\tilde{T}(a)\tilde{T}(b)\tilde{\xi}, \tilde{T}(c)\tilde{\xi}),$$

that is, $(UT(a)T(b)\xi, UT(c)\xi) = (\tilde{T}(a)UT(b)\xi, UT(c)\xi)$. Therefore $UT(a) = \tilde{T}(a)U$, so $\operatorname{Ad} U$ defines the required isomorphism.

In the general case the proof is basically the same, but instead of one state ψ we have to choose a faithful family of states on A and consider direct sums of the GNS-representations defined by the corresponding positive linear functionals on A_ω and $\tilde{A}_\omega$. □

3.3 Bundle structure and RKK-fibration

Recall that we are considering $G = V = \mathbb{R}^d = \mathbb{R}^{2n}$ and 2-cocycle $\omega_h(x, y) = e^{i\frac{h}{2}\langle x, Jy\rangle}$ where $J \in M_d(\mathbb{R})$ is skew-symmetric and $h \in [0, 1]$. We shall denote $\omega = \omega_1$. Assume we have a strongly continuous action $\alpha : \mathbb{R}^d \longrightarrow Aut(A)$ on a separable C*-algebra A. Then for each $h \in [0, 1]$ we shall consider the action $\alpha^h : \mathbb{R}^d \longrightarrow Aut(A)$ given by $\alpha^h_x = \alpha_{\sqrt{h}x}$ for $x \in \mathbb{R}^d$. We define

$$A_h = A_{\omega, \alpha^h}$$

to be the deformation of A with respect to the action α^h and 2-cocycle ω.

Lemma 3.3.1. *There is a canonical equivariant isomorphism $\theta_h : A_h \longrightarrow A_{\omega_h}$.*

Proof. The oscillatory integral (3.2.1) used in defining the product in A_h is

$$\begin{aligned} a *_1 b &= \frac{1}{\pi^{2n}} \int_{V\times V} \alpha^h_x(a)\alpha^h_y(b)e^{-2i\langle x,Jy\rangle}\,dx\,dy \\ &= \frac{1}{\pi^{2n}} \int_{V\times V} \alpha_{\sqrt{h}x}(a)\alpha_{\sqrt{h}y}(b)e^{-2i\langle x,Jy\rangle}\,dx\,dy \end{aligned}$$

which after applying a change of variables $v = \sqrt{h}x$ and $w = \sqrt{h}y$ becomes

$$\begin{aligned} a *_1 b =& \frac{1}{\pi^{2n}}\frac{1}{h^{2n}} \int_{V\times V} \alpha_v(a)\alpha_w(b)e^{-2i\langle \frac{1}{\sqrt{h}}v,J\frac{1}{\sqrt{h}}w\rangle}\,dv\,dw \\ =& \frac{1}{(\pi h)^{2n}} \int_{V\times V} \alpha_v(a)\alpha_w(b)e^{\frac{-2i}{h}\langle v,Jw\rangle}\,dv\,dw, \end{aligned}$$

and the latter integral is used in defining the product in A_{ω_h}. This shows the canonical mapping between the two algebras. ☐

Let $B = C([0,1]) \otimes A = C([0,1], A)$ be equipped with the obvious $C([0,1])$-algebra structure $\Phi_B : C([0,1]) \longrightarrow ZM(B)$, $\Phi_B(f)(g\otimes a) = fg\otimes a$. Define the action

$$\beta : \mathbb{R}^d \longrightarrow Aut(B), \quad \beta_x(y)(h) = \sigma_{\sqrt{h}x}(y(h)), \tag{3.3.1}$$

for $x \in \mathbb{R}^n$, $y \in B$, $h \in [0,1]$. We then get the deformed algebra B_ω with an action β^ω according to Theorem 2.2.1. Then B_ω is also a $C([0,1])$-algebra. In the following result, the fact that B_ω is a continuous C*-bundle, was proved by Rieffel ([52], see also [55] for a brief presentation).

Theorem 3.3.2. *The $C([0,1])$-algebra B_ω is a continuous C*-bundle with fibers isomorphic to A_{ω_h}. The isomorphism between fibers $(B_\omega)_h \cong A_{\omega_h}$ is given by the composition of the homomorphisms $B_\omega \hookrightarrow M(B\rtimes_\beta\mathbb{R}^d)$ and $ev_h\rtimes\mathbb{R}^d : M(B\rtimes\mathbb{R}^d) \longrightarrow M(A\rtimes_{\alpha^h}\mathbb{R}^d)$ which gives us $(B_\omega)_h \cong A_h$, which is finally followed by the isomorphism θ_h.*

Proposition 3.3.3. *B_ω is an RKK-fibration.*

Proof. It follows from Theorem 1.4.8 that B_ω is $RKK([0,1];\cdot,\cdot)$-equivalent, with dimension shift $n\,(mod\,2)$, to $B_\omega \rtimes_{\beta^\omega} \mathbb{R}^n$. By the isomorphism $B\rtimes_\beta\mathbb{R}^n \cong B_\omega\rtimes_{\beta^\omega}\mathbb{R}^n$ the latter algebra is $RKK([0,1];\cdot,\cdot)$-equivalent to $B\rtimes_\beta\mathbb{R}^n$, which by Theorem 1.4.8 again is $RKK([0,1];\cdot,\cdot)$-equivalent, with another dimension shift $n\,(mod\,2)$, to $B = C([0,1])\otimes A$. The total dimension shift thus far is $2n\,(mod\,2) = 0$, i.e. the net effect being no dimension shift, so B_ω is plainly $RKK([0,1];\cdot,\cdot)$-equivalent to B. Finally, the algebra $B = C([0,1])\otimes A$ is clearly an RKK-fibration (Remark 1.4.11), thus proving the claim. ☐

It follows from Proposition 3.3.3 and Lemma 1.4.12 that B_ω is a KK-fibration. Taking the identity function of the 1-simplex, $f : \Delta^1 = [0,1] \longrightarrow [0,1]$, $f(s) = s$, we conclude that the evaluation map $q_s : B_\omega \longrightarrow (B_\omega)_s$ is a KK-equivalence.

Note that an isomorphism of K-theory was also established in section 2.3. We now summarize the above discussion in the following result.

Theorem 3.3.4. *The collection of C*-algebras $\{A_{\omega_h}\}_{h\in[0,1]}$ forms a continuous field, and for every $h \in [0,1]$ the evaluation map ev_h is a KK-equivalence.*

3.4 Theta deformation

Here we discuss a special case of Rieffel deformation, namely *theta deformation* and one possible variation to the above approach to KK-equivalence by bundle methods. Theta deformation concerns a separable C*-algebra A on which there is a strongly continuous action of the n-torus, $\sigma : \mathbb{T}^n \longrightarrow Aut(A)$, with a given skew-symmetric matrix $\theta \in M_n(\mathbb{R})$. This is just a special case of Rieffel deformation in which the n-torus is regarded as the quotient $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, and one obtains the deformed algebra A_θ. Of course, this setup is nothing other than our cocycle deformation as explained in Example 2.1.2 for $G = \mathbb{Z}^n = \widehat{\mathbb{T}^n}$ with 2-cocycle $\omega(k,m) = e^{\pi i k\cdot\theta m}$.

Theta deformation was introduced by Connes, Dubois-Violette and Landi ([16], [17]), who considered the Rieffel deformation of the *-algebra $C^\infty(M)$ of smooth functions on a compact Riemannian spin manifold equipped with an action of $\mathbb{T}^n$. Connes, Dubois-Violette and Landi arrive at the deformed algebra in a different way, namely as the fixed point subalgebra of a diagonal action which is essentially equivalent to Rieffel's picture with regards to spectral subspace decomposition. We can also here mention Matsumoto's investigations [39] on noncommutative spheres, with relations to irrational rotation algebras.

The bundle evaluation KK-isomorphism from the previous section certainly applies to theta deformations. For theta deformations, however, the existence of a KK-isomorphism has been observed by several authors (see e.g. [61]). The shortest, but not the easiest, route to such an isomorphism is by using the strong Baum-Connes conjecture for $\mathbb{Z}^n$, whose validity follows from the PV-sequence in KK-theory. It is not surprising then, that a KK-isomorphism of theta deformations can be deduced using only the PV-sequence. Our goal in this section is to present such an argument. The idea is to define a continuous C*-bundle over $[0,1]$ whose fiber over $t \in [0,1]$ is not $A_{t\theta}$ per se, but an iterated crossed product by the integers which is strongly Morita equivalent to $A_{t\theta}$. The PV-sequence is then applied to this crossed product bundle to conclude that the bundle evaluation map has a KK-contractible kernel.

We will now recall the definition of the theta deformation. Let A be a separable C*-algebra with a strongly continuous action $\sigma : \mathbb{T}^n \longrightarrow Aut(A)$. The twisted group C*-algebra $C^*(\mathbb{Z}^n, \omega)$, which is denoted $C(\mathbb{T}^n_\theta)$ in [16] and [17], is the universal unital C*-algebra generated by unitaries u_j, $j = 1, \ldots, n$, satisfying $u_j u_k = e^{2\pi i\theta_{j,k}} u_k u_j$. The dual action τ is defined by $\tau_s(u_j) = e^{2\pi i s_j} u_j$, for $s \in \mathbb{T}^n$, $j = 1, \ldots, n$. The theta deformation A_θ is then by definition $A_\theta = (A \otimes C(\mathbb{T}^n_\theta))^{\sigma\otimes\tau^{-1}}$. We may use the following spectral subspace decomposition to describe A_θ in terms of series of elements of A, with the deformed product denoted $\times_\theta$. Let $s \in \mathbb{T}^n$, $s = (e^{2\pi i s_1}, \ldots, e^{2\pi i s_n})$. Each $r \in \mathbb{Z}^n$, $r = (r_1, \ldots, r_n)$ where $r_j \in \mathbb{Z}$, corresponds to a character $s \mapsto r(s) = e^{2\pi i r\cdot s}$, where $r \cdot s = \sum_j r_j s_j$. That is, $\mathbb{Z}^n \cong \widehat{\mathbb{T}^n}$ is the Pontryagin dual group. The r-th spectral subspace, where $r \in \mathbb{Z}^n$, for the action is

$$A_r = \{a \in A : \sigma_s(a) = e^{2\pi i r\cdot s} a, \text{ for every } s \in \mathbb{T}^n\}. \tag{3.4.1}$$

The deformed product $\times_\theta$ takes the following form on the spectral subspaces; for $a_{r_1} \in A_{r_1}$ and $b_{r_2} \in A_{r_2}$, $r_1, r_2 \in \mathbb{Z}^n$, we have

$$a_{r_1} \times_\theta b_{r_2} = e^{\pi i\, r_1\cdot\theta r_2} a_{r_1} b_{r_2}.$$

The product extends linearly to elements decomposed as norm convergent series

$a = \sum_{r_1 \in \mathbb{Z}^n} a_{r_1}$ and $b = \sum_{r_2 \in \mathbb{Z}^n} b_{r_2}$ by

$$a \times_\theta b = \sum_{r_1, r_2} a_{r_1} \times_\theta b_{r_2} = \sum_{r_1, r_2} e^{\pi i r_1 \cdot \theta r_2} a_{r_1} b_{r_2}.$$

Denote by $\mathcal{A}_\theta$ the subalgebra of A_θ consisting of finite series $\sum_{r \in \mathbb{Z}^n} a_r$.

Define a 2-cocycle ω on $\mathbb{Z}^n = \widehat{\mathbb{T}^n}$ by $\omega(k, m) = e^{\pi i k \cdot \theta m}$. Then we can identify $C(\mathbb{T}^n_\theta)$ with $C^*(\mathbb{Z}^n, \omega)$ by mapping u_j into $\lambda^\omega_{e_j}$, where e_j is the j-th element of the canonical basis of $\mathbb{Z}^n$. From our discussion in Section 3.1 or Example 2.1.2 it is clear that under this identification A_θ coincides with the ω-deformation A_ω of A. Finally, as we mentioned above, theta deformations are particular cases of Rieffel deformations, or in other words, of cocycle deformations by cocycles on $\mathbb{R}^n$. Namely, identifying $\mathbb{T}^n$ with $\mathbb{R}^n/\mathbb{Z}^n$ and denoting by q the quotient map $\mathbb{R}^n \longrightarrow \mathbb{T}^n$, we can view the action σ as an action of $G = \mathbb{R}^n$. We identify $\widehat{G}$ with G using the pairing $(x, y) \mapsto e^{2\pi i x \cdot y}$. Consider the cocycle $\omega(x, y) = e^{\pi i x \cdot \theta y}$ on $G = \mathbb{R}^n$. The deformation A_ω as described in section 3.1 coincides with the theta deformation A_θ. Indeed, clearly $\mathcal{A}_\omega = \mathcal{A}_\theta$, and we only have to check that the norms are the same. This follows from the fact that the natural representation of $C(\mathbb{T}^n_\theta) = C^*(\mathbb{Z}^n, \omega_{|\mathbb{Z}^n}) \longrightarrow M(C^*(\mathbb{R}^n, \omega))$ is injective, as we can find a copy of the regular representation of $C^*(\mathbb{Z}^n, \omega_{|\mathbb{Z}^n})$ on $l^2(\mathbb{Z}^n)$ in the representation of this algebra on $L^2(\mathbb{R}^n)$.

Lemma 3.4.1. $A_\theta \sim_M A \rtimes_\sigma \mathbb{T}^n \rtimes_{\gamma_1} \mathbb{Z} \rtimes \cdots \rtimes_{\gamma_n} \mathbb{Z}$.

Proof. There is a strong Morita equivalence (see e.g. [48])

$$(A \otimes C(\mathbb{T}^n_\theta))^{\sigma \otimes \tau^{-1}} \sim_M (A \otimes C(\mathbb{T}^n_\theta)) \rtimes_{\sigma \otimes \tau^{-1}} \mathbb{T}^n.$$

The latter crossed product algebra is *-isomorphic to the crossed product in the statement of the lemma, which we now define. Inside the crossed product $(A \otimes C(\mathbb{T}^n_\theta)) \rtimes_{\sigma \otimes \tau^{-1}, r} \mathbb{T}^n$ we have C*-algebras A_j generated by $(A \otimes 1) \rtimes_{\sigma \otimes \tau^{-1}, r} \mathbb{T}^n$ and $1 \otimes u_k$ for $k \leqslant j$. Conjugation by $1 \otimes u_j$ defines the automorphism $\gamma_j \in Aut(A_{j-1})$, and the crossed product $A_{j-1} \rtimes_{\gamma_j, r} \mathbb{Z}$ can be identified with A_j. This gives the required iterated crossed product decomposition. □

Now we consider $B = C([0,1]) \otimes A \rtimes_\sigma \mathbb{T}^n$. Replacing θ with $h\theta$, for $h \in [0,1]$ we also get actions γ^h_j in the context of the preceding proof. These actions assemble into actions α_j on B in similar form as in the lemma, and we get an iterated crossed product $B \rtimes_{\alpha_1} \mathbb{Z} \rtimes \cdots \rtimes_{\alpha_n} \mathbb{Z}$. If we denote by $v_1, \ldots v_n$ the implementing unitaries of the respective actions $\alpha_1, \ldots, \alpha_n$, i.e. $\alpha_j(x) = v_j x v_j^*$ inside the crossed product, then explicitly we have

$$\alpha_k((f \otimes g) v_j^m) = v_k((f \otimes g) v_j^m) v_k^* = (h^m_{j,k} f \otimes \hat{\sigma_k}^m(g)) v_j^m, \quad \text{for } j, k = 1, \ldots, n, \tag{3.4.2}$$

and $m \in \mathbb{Z}$, where $h_{j,k} \in C([0,1])$ is the function $h_{j,k}(t) = e^{2\pi i t \theta_{j,k}}$.

Lemma 3.4.2. *$B \rtimes_{\alpha_1} \mathbb{Z} \rtimes \cdots \rtimes_{\alpha_n} \mathbb{Z}$ is a continuous C^*-bundle with fibers $(A \rtimes \mathbb{T}^n) \rtimes \mathbb{Z} \rtimes \cdots \rtimes \mathbb{Z}$.*

Proof. For $h \in [0,1]$, let $\pi_h : B \rtimes_{1 \otimes \sigma} \mathbb{T}^n \longrightarrow A \rtimes_\sigma \mathbb{T}^n$ be the canonical evaluation map. Then $\pi_h \circ \alpha_1 = \gamma^h_1 \circ \pi_h$ and so induces a *-homomorphism between crossed products

$$\pi_h : (C([0,1]) \otimes A \rtimes_\sigma \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z} \longrightarrow A \rtimes_\sigma \mathbb{T}^n \rtimes_{\gamma^h_1} \mathbb{Z},$$

which is a continuous C*-bundle by Lemma 1.4.5. Iterating this, we have $\pi_h \circ \alpha_j = \gamma_j^h$, where we still denote by π_h the induced *-homomorphism at each step. We thus get the continuous C*-bundle

$$\pi_h : C([0,1], A \rtimes_\sigma \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z} \rtimes \cdots \rtimes_{\alpha_n} \mathbb{Z} \longrightarrow A \rtimes_\sigma \mathbb{T}^n \rtimes_{\gamma_1^h} \mathbb{Z} \rtimes \cdots \rtimes_{\gamma_n^h} \mathbb{Z}. \quad (3.4.3)$$

□

For each $h \in [0,1]$ let $I_h = \{f \in C([0,1]) \mid \quad f(h) = 0\}$ be the ideal of functions vanishing at the point $h \in [0,1]$. The kernel of the *-homomorphism π_h in (3.4.3) is

$$ker\, \pi_h = (I_h \otimes A \rtimes_\sigma \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z} \rtimes \cdots \rtimes_{\alpha_n} \mathbb{Z}.$$

Using the fact that I_h is contractible, we get that $I_h \otimes A \rtimes_\sigma \mathbb{T}^n$ is also contractible.

We recall a few general facts which we will appeal to shortly, in particular contractibility of cones and the Pimsner-Voiculescu six-term exact sequence. First recall from section 1.2 that a C*-algebra B is called *KK-contractible* if $KK(B,B) = 0$. This also implies $KK(B,D) = 0 = KK(D,B)$ for any other C*-algebra D.

Suppose there is an action $\beta \in Aut(B)$. Recall the Pimsner-Voiculescu six-term exact sequence from (1.2.5)

$$\begin{array}{ccccc} KK(D,B) & \xrightarrow{1-\beta_*} & KK(D,B) & \longrightarrow & KK(D, B \rtimes_\beta \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ KK^1(D, B \rtimes_\beta \mathbb{Z}) & \longleftarrow & KK^1(D,B) & \xleftarrow[1-\beta_*]{} & KK^1(D,B) \end{array}$$

Observe that if B is KK-contractible, then the six-term exact sequence reads

$$\begin{array}{ccccc} 0 & \xrightarrow{1-\beta_*} & 0 & \longrightarrow & KK(D, B \rtimes_\beta \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ KK^1(D, B \rtimes_\beta \mathbb{Z}) & \longleftarrow & 0 & \xleftarrow[1-\beta_*]{} & 0 \end{array}$$

and using in particular $D = B \rtimes_\beta \mathbb{Z}$ we deduce $KK(B \rtimes_\beta \mathbb{Z}, B \rtimes_\beta \mathbb{Z}) = 0$, i.e. $B \rtimes_\beta \mathbb{Z}$ is KK-contractible. Recall also that given any separable C*-algebra D, its cone $Cone(D) = C_0([0,1)) \otimes D$ is KK-contractible.

Theorem 3.4.3. *For every $h \in [0,1]$ the bundle map*

$$\pi_h : C([0,1], A \rtimes_\sigma \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z} \rtimes \cdots \rtimes_{\alpha_n} \mathbb{Z} \longrightarrow A \rtimes_\sigma \mathbb{T}^n \rtimes_{\gamma_1^t} \mathbb{Z} \rtimes \cdots \rtimes_{\gamma_n^h} \mathbb{Z}$$

gives a KK-equivalence.

Proof. As $I_h \otimes A \rtimes_\sigma \mathbb{T}^n$ is contractible, it is KK-contractible. A repeated Pimsner-Voiculescu six-term sequence argument as above establishes that $ker\, \pi_h = (I_h \otimes A \rtimes_\sigma \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z} \rtimes \cdots \rtimes_{\alpha_n} \mathbb{Z}$ is KK-contractible. This implies that π_h gives a KK-equivalence element. □

Theorem 3.4.4. *The evaluation map of the continuous field $\Gamma((A_{h\theta})_{h\in[0,1]})$*

$$\pi_h : \Gamma((A_{h\theta})_{h\in[0,1]}) \longrightarrow A_{h\theta}$$

is a KK-equivalence

Proof. The crossed product $A \rtimes_\sigma \mathbb{T}^n \rtimes_{\gamma_1^h} \mathbb{Z} \rtimes \cdots \rtimes_{\gamma_n^h} \mathbb{Z}$ is strongly Morita equivalent to $A_{h\theta}$ by Lemma 3.4.1. On the other hand the crossed product $C([0,1], A \rtimes_\sigma \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z} \rtimes \cdots \rtimes_{\alpha_n} \mathbb{Z}$ can be taken to be the bundle algebra (of a continuous field of crossed products) $\Gamma = \Gamma((A_{h\theta})_{h \in [0,1]})$. We abuse notation here slightly, as the bundle algebra Γ does not have fibers $A_{h\theta}$ per se, but fibers that are strongly Morita equivalent to $A_{h\theta}$. The result follows by combining the KK-equivalence from Theorem 3.4.3 with the KK-equivalence element induced by the strong Morita equivalence. $\square$

Part III

Index theory

Chapter 4

Index theory of theta deformations

In this chapter we show the invariance of the index pairing under theta deformation. We use the KK-equivalence of Theorem 3.3.4 (or Theorem 3.4.4, as both theorems state the same result with different proofs). We also show an isomorphism of the periodic cyclic cohomology groups of the original and deformed algebra which is compatible with the isomorphism in K-theory. This was also done in [61], but we give a different and more direct argument using the isomorphism of Theorem 2.2.1.

4.1 Deformation of spectral triples

We are working with a C*-algebra A with an action $\sigma : \mathbb{T}^n \longrightarrow Aut(A)$. Let $\mathcal{A}^\infty \subseteq A$ denote the set of smooth elements for the action. By using the spectral subspace decomposition $a = \sum_r a_r$ (see eq. (3.4.1)), we can define a family of seminorms $||\cdot||_k$, $k \in \mathbb{N}$, on $\mathcal{A}^\infty$ by $||a||_k = \sum_r ||a_r||(1 + |r|)^k$. This makes $\mathcal{A}^\infty$ a Frechet *-algebra. As explained in the discussion following eq. (3.4.1), we can define the deformed product $\times_\theta$ on $\mathcal{A}^\infty$ by $a_r \times_\theta b_p = e^{\pi i r\cdot\theta p} a_r b_p$, which is extended linearly to give

$$a \times_\theta b = \sum_{r,p} e^{\pi i r\cdot\theta p} a_r b_p$$

for $a = \sum_{r\in\mathbb{Z}^n} a_r$, $b = \sum_{p\in\mathbb{Z}^n} b_p \in \mathcal{A}^\infty$. We thus get the deformed Frechet algebra $\mathcal{A}^\infty_\theta \subset A_\theta$.

Now suppose we have a $\mathbb{T}^n$-equivariant spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, where $\varphi : \mathcal{A}^\infty \longrightarrow B(\mathcal{H})$ is the *-representation, the action σ is unitarily implemented by $U : \mathbb{T}^n \longrightarrow B(\mathcal{H})$, $\varphi(\sigma_s(a)) = U_s\varphi(a)U_s^*$ for $s \in \mathbb{T}^n$, $a \in \mathcal{A}^\infty$, and $U_s D = DU_s$ for all $s \in \mathbb{T}^n$. As usual we let $q : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$ denote the quotient map.

Proposition 4.1.1. *There is a *-representation $\varphi_\theta : \mathcal{A}^\infty_\theta \longrightarrow B(\mathcal{H})$ given by*

$$\varphi_\theta(a) = \sum_r \varphi(a_r) U_{q(-\frac{\theta}{2}r)}$$

for $a = \sum_r a_r \in \mathcal{A}^\infty$.

Proof. Recall from Lemma 3.1.1 and the discussion following it, the embedding $A_\theta \hookrightarrow M(A \rtimes_\sigma \mathbb{T}^n)$, $a_r \mapsto a_r\lambda_{q(-\frac{\theta}{2})}$, which basically allows one to describe A_θ as the algebra generated by elements of the form $a_r\lambda_{q(-\frac{\theta}{2})}$. At the level of $\mathcal{A}^\infty_\theta \subset A_\theta$, the representation φ_θ coincides with this embedding, that is, the restriction of the representation of the crossed product (extended to the multiplier algebra) to the deformed algebra. □

We will assume without too much loss of generality that we get a spectral triple $(\mathcal{A}_\theta^\infty, \mathcal{H}, D_\theta)$ where $D_\theta = D$. In general, one may have to restrict to a smaller algebra ([61]).

4.2 Invariance of the index

Here we comment on the index pairing in the general case of a theta deformation A_θ using the KK-equivalence of Theorem 3.4.4. The index pairing is the pairing between K-theory and K-homology

$$K_0(A) \times K^0(A) \longrightarrow \mathbb{Z}$$

$$\langle [e], [(\mathcal{H}, F)] \rangle = index\left(e(F^+ \otimes 1_k)e : e\mathcal{H}^k \longrightarrow e\mathcal{H}^k \right), \tag{4.2.1}$$

for a projection $e \in M_k(A)$ and Fredholm module $(\mathcal{H}, F)$ for A. This pairing is nothing but the KK-product

$$KK(\mathbb{C}, A) \times KK(A, \mathbb{C}) \longrightarrow KK(\mathbb{C}, \mathbb{C}) \tag{4.2.2}$$

after the identifications $K_0(A) = KK(\mathbb{C}, A)$, $K^0(A) = KK(A, \mathbb{C})$ and $KK(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$.

See also [61] for a discussion of theta deformation and the invariance of the index, and moreover a calculation of the Chern character map for the deformation.

Given the even spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ there is the associated Fredholm module $(\mathcal{H}, F)$ with $F = D(1+D^2)^{-\frac{1}{2}}$. We shall denote by $[D] = [(\mathcal{H}, \varphi, F)] \in K^0(A)$ the corresponding element of K-homology. Likewise we denote by $[D_\theta] = [(\mathcal{H}, \varphi_\theta, F)] \in K^0(A_\theta)$ the element associated to the spectral triple $(\mathcal{A}_\theta^\infty, \mathcal{H}, D_\theta)$.

Corollary 4.2.1. *The KK-equivalence of Theorem* 3.3.4 *(or Theorem* 3.4.4*) induces an isomorphism $K^0(A) \cong K^0(A_\theta)$ mapping $[D] \mapsto [D_\theta]$.*

Proof. Let $\Gamma = \Gamma((A_{t\theta})_{t\in[0,1]})$. From the bundle maps $\pi_0 : \Gamma \longrightarrow A$ and $\pi_1 : \Gamma \longrightarrow A_\theta$ we get by Theorem 3.3.4 (or Theorem 3.4.4) the KK-equivalence elements $[\pi_0] \in KK(\Gamma, A)$ and $[\pi_1] \in KK(\Gamma, A_\theta)$. The relevant mappings between KK-groups is described by the KK-products

$$\begin{tikzcd}
 & KK(\Gamma, \mathbb{C}) & \\
KK(A, \mathbb{C}) \arrow[ur, "{[\pi_0]\cdot}"] \arrow[rr, dashed, "{[\pi_1]^{-1} \cdot [\pi_0]\cdot}"'] & & KK(A_\theta, \mathbb{C}) \arrow[ul, "{[\pi_1]\cdot}"']
\end{tikzcd}$$

where $[\pi_0] = [(A, \pi_0, 0)] \in KK(A, \mathbb{C})$ and $[\pi_1] = [(A_\theta, \pi_1, 0)] \in KK(\Gamma, A_\theta)$ are the KK-cycle descriptions.

The element $[D] = [(\mathcal{H}, \varphi, F)] \in KK(A, \mathbb{C})$ is the element canonically associated to the given spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ as explained above, and upon taking the KK-product we get

$$[\pi_0] \cdot [D] = [(\mathcal{H}, \varphi \circ \pi_0, F)] \in KK(\Gamma, \mathbb{C}). \tag{4.2.3}$$

Likewise $[D_\theta] = [(\mathcal{H}, \varphi_\theta, F)] \in KK(A_\theta, \mathbb{C})$ is the element associated to the deformed spectral triple $(\mathcal{A}_\theta^\infty, \mathcal{H}, D_\theta)$, and the KK-product is then

$$[\pi_1] \cdot [D_\theta] = [(\mathcal{H}, \varphi \circ \pi_1, F)] \in KK(\Gamma, \mathbb{C}). \tag{4.2.4}$$

It will be enough to establish the equality $[\pi_0] \cdot [D] = [\pi_1] \cdot [D_\theta]$ in $KK(\Gamma, \mathbb{C})$. This follows from homotopy of KK-cycles. Indeed, let $(E, \phi, F) \in KK(\Gamma, I\mathbb{C})$ be the element where $E = C([0,1], \mathcal{H})$, $\phi : \Gamma \longrightarrow \mathcal{L}_{I\mathbb{C}}(E)$, $(\phi(s)\xi)(t) = s(t)\xi(t)$, and $I\mathbb{C} = C([0,1])\otimes\mathbb{C} = C([0,1])$. Let ev_0 and ev_1 denote the respective evaluation morphisms $E \longrightarrow \mathcal{H}$. It is easy to check (using details explained in [17]) that (E, ϕ, F) provides a homotopy between the KK-cycles (4.2.3) and (4.2.4), i.e. isomorphisms of the KK-cycles with the pushouts of ev_0 and ev_1 respectively,

$$(E_{ev_0}, \phi_{ev_0}, F_{ev_0}) \cong [(\mathcal{H}, \varphi \circ \pi_0, F)] \quad \text{and} \quad (E_{ev_1}, \phi_{ev_1}, F_{ev_1}) \cong [(\mathcal{H}, \varphi \circ \pi_1, F)].$$

□

The KK-equivalence implies the isomorphisms

$$K_0(A) = KK(\mathbb{C}, A) \longrightarrow KK(\mathbb{C}, A_\theta) = K_0(A_\theta), \quad [e] \longmapsto [e] \cdot [\pi_0]^{-1} \cdot [\pi_1],$$

and

$$K^0(A) = KK(A, \mathbb{C}) \longrightarrow KK(A_\theta, \mathbb{C}) = K^0(A_\theta), \quad [(\mathcal{H}, F)] \longmapsto [\pi_1]^{-1} \cdot [\pi_0] \cdot [(\mathcal{H}, F)],$$

and regarding the index pairing (4.2.1) or equivalently the KK-product (4.2.2), we get

$$\begin{array}{ccc} K_0(A) \times K^0(A) & \xrightarrow{\text{index}} & \mathbb{Z} \\ \Big\downarrow & & \Big\downarrow \| \\ K_0(A_\theta) \times K^0(A_\theta) & \xrightarrow[\text{index}]{} & \mathbb{Z} \end{array}$$

where $[e] \cdot [(\mathcal{H}, F)]$ is the top index pairing and

$$[e] \cdot [\pi_0]^{-1} \cdot [\pi_1] \cdot [\pi_1]^{-1} \cdot [\pi_0] \cdot [(\mathcal{H}, F)] = [e] \cdot [(\mathcal{H}, F)]$$

is the bottom index pairing after having followed the isomorphisms induced by the KK-equivalences.

4.3 Isomorphism of periodic cyclic cohomology groups

An isomorphism was between the periodic cyclic cohomology groups of A and A_θ which was also compatible with the K-theory isomorphism was shown in [61], considering an action of $\mathbb{T}^2$. In [63] rigidity of the periodic cyclic cohomology for noncommutative tori was shown using Getzler's Gauss-Manin connection.

In this section we shall establish an isomorphism between periodic cyclic cohomology groups of A and its theta deformation A_θ by an action of $\mathbb{T}^n$. In the following we consider essentially the isomorphism $A \rtimes_\alpha \mathbb{T}^n \longrightarrow A_\theta \rtimes_{\alpha^\theta} \mathbb{T}^n$, but we lift the $\mathbb{T}^n$-action to an $\mathbb{R}^n$-action and so consider crossed products by $\mathbb{R}^n$ instead.

This is important as the Elliott-Natsume-Nest ([23]) isomorphism in periodic cyclic cohomology concerns crossed products by $\mathbb{R}^n$, a result we shall combine with its counterpart in K-theory, the Connes-Thom isomorphism.

We briefly recall the smooth crossed product. We have our Frechet algebra $\mathcal{A}^\infty$ with seminorms $||\cdot||_k$, and the smooth action $\alpha : \mathbb{R}^n \longrightarrow Aut(\mathcal{A}^\infty)$. On the function space $\mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty)$ of $\mathcal{A}^\infty$-valued smooth rapidly decreasing functions with pointwise multiplication, we have the topology given by seminorms

$$||f||_{k,m} = \sup_{t\in\mathbb{R}^n}(1+|t|^2)^{\frac{k}{2}}||\frac{\partial^m}{\partial t^m}f(t)||_k$$

for $k \geqslant 0$ and multi-indices m, where $|\cdot|$ is the Euclidean norm. The convolution product on $\mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty)$ is defined as usual by

$$(f \times g)(t) = \int_{\mathbb{R}^n} f(s)\alpha_s(g(t-s))\, ds.$$

The smooth crossed product $\mathcal{A}^\infty \rtimes_\alpha \mathbb{R}^n$ is by definition the Frechet algebra $\mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty)$ with the above convolution product and seminorms $||\cdot||_{k,m}$.

Proposition 4.3.1. *The isomorphism $\mathcal{A}^\infty \rtimes_\alpha \mathbb{R}^n \cong \mathcal{A}^\infty_\theta \rtimes_{\alpha^\theta} \mathbb{R}^n$ maps $f \mapsto \widetilde{f}$ where*

$$\widetilde{f}(t) = \sum_{r\in\mathbb{Z}^n} f_r(t + \frac{\theta r}{2}).$$

Proof. Note that the mapping is merely an equivalent way of expressing the componentwise mapping $a_\chi \mapsto a_\chi \lambda_{-r(\chi)}$ established in the discussion of Lemma 3.1.1. We want to show that the mapping is an isomorphism of smooth crossed products $\mathcal{A}^\infty \rtimes_\alpha \mathbb{R}^n \longrightarrow \mathcal{A}^\infty_\theta \rtimes_{\alpha^\theta} \mathbb{R}^n$ We claim that for every k there exist $C > 0$ such that $||\widetilde{f}||_{k,m} \leqslant C||f||_{l,m}$ for all m, where $l = 2k + n + 1$. Since the map $f \mapsto \widetilde{f}$ commutes with partial derivatives, it is enough to consider $m = 0$. Take f such that $||f||_{l,0} = 1$. We have to find an upper bound on $||\widetilde{f}||_{k,0}$ that is independent of f. By assumption, we have

$$||f_r(t)|| \leqslant (1+|t|)^{-l}(1+|r|)^{-l}.$$

It follows that $||\widetilde{f}||_{k,0}$ is not larger than the supremum over t of the expressions

$$\sum_r (1+|t|)^k(1+|t-\frac{\theta r}{2}|)^{-l}(1+|r|)^{-l}.$$

Fix t and divide the above sum into two parts, with $|\theta r| \leqslant |t|$ and with $|\theta r| > t$. In the first part we have $|t - \frac{\theta r}{2}| \geqslant \frac{|t|}{2}$ and so it is bounded by

$$\sum_{|\theta r|\leqslant|t|}(1+|t|)^k(1+\frac{|t|}{2})^{-l}(1+|r|)^{-(l-k)} \leqslant 2^l \sum_{r\in\mathbb{Z}^n}(1+|r|)^{-(k+n+1)} < \infty,$$

where we used that $(1+\frac{|t|}{2})^{-l} \leqslant 2^l(1+|t|)^{-l}$. On the other hand, the second part is bounded by

$$\sum_{|\theta r|>|t|}(1+|\theta r|)^k(1+|r|)^{-(l-k)} \leqslant (1+||\theta||)^k \sum_{r\in\mathbb{Z}^n}(1+|r|)^{-(n+1)} < \infty,$$

where we used that $(1+|\theta r|)^k \leqslant (1+||\theta||)^k(1+|r|)^k$. $\square$

We consider cyclic cohomology of Frechet algebras, in which case the cochains are assumed to be continuous.

Theorem 4.3.2. *There is an isomorphism $HP^*(\mathcal{A}_\theta^\infty) \cong HP^*(\mathcal{A}^\infty)$ which is compatible with the isomorphism $K_*(\mathcal{A}_\theta^\infty) \cong K_*(\mathcal{A}^\infty)$.*

Proof. The theorem of Elliott-Natsume-Nest ([23]) provides an isomorphism

$$\sharp_\alpha : HP^*(\mathcal{A}^\infty) \longrightarrow HP^*(\mathcal{A}^\infty \rtimes_\alpha \mathbb{R}^n)$$

which satisfies $\langle \varphi, x \rangle = \langle \sharp_\alpha(\varphi), \Psi_\alpha(x) \rangle$, for $\varphi \in HP^*(\mathcal{A}^\infty)$, $x \in K_*(\mathcal{A}^\infty)$, where

$$\Psi_\alpha : K_*(\mathcal{A}^\infty) \longrightarrow K_*(\mathcal{A}^\infty \rtimes_\alpha \mathbb{R}^n)$$

is the Connes-Thom isomorphism. Let $T^* : HP^*(\mathcal{A}^\infty \rtimes_\alpha \mathbb{R}^n) \longrightarrow HP^*(\mathcal{A}_\theta^\infty \rtimes_{\alpha^\theta} \mathbb{R}^n)$ and $T_* : K_*(\mathcal{A}^\infty \rtimes_\alpha \mathbb{R}^n) \longrightarrow K_*(\mathcal{A}_\theta^\infty \rtimes_{\alpha^\theta} \mathbb{R}^n)$ denote the respective isomorphisms induced by the isomorphism of the preceding proposition. Clearly we have $\langle T^*(c), T_*(d) \rangle = \langle c, d \rangle$ for any $c \in HP^*(\mathcal{A}^\infty \rtimes_\alpha \mathbb{R}^n)$, $d \in K_*(\mathcal{A}^\infty \rtimes_\alpha \mathbb{R}^n)$. It follows that

$$\langle T^*(\sharp_\alpha(\varphi)), T_*(\Psi_\alpha(x)) \rangle = \langle \sharp_\alpha(\varphi), \Psi_\alpha(x) \rangle = \langle \varphi, x \rangle,$$

which shows the invariance of the pairing. □

Chapter 5

Local index formula for theta deformations of manifolds

We give a formula for the Chern character of the spectral triple of a spin manifold with a $\mathbb{T}^n$-action, in terms of the $\mathbb{T}^n$-equivariant Chern character. We use the results on the equivariant JLO-cocycle and the equivariant Chern character from section 1.3.

5.1 Local index formula

Let M be a compact Riemannian spin manifold (of dimension m) equipped with an action of $\mathbb{T}^n$ ($n \geqslant 2$) by isometries, $\mathbb{T}^n \subseteq Iso(M)$. This $\mathbb{T}^n$-action on M will be expressed $x \mapsto sx$ for $x \in M$ and $s \in \mathbb{T}^n$. The induced $\mathbb{T}^n$-action on $C(M)$ will be denoted

$$\sigma : \mathbb{T}^n \longrightarrow Aut(C(M)), \sigma_s(f)(x) = f(s^{-1}x), f \in C(M), x \in M. \tag{5.1.1}$$

Let $\theta \in Mat_n(\mathbb{R})$ be a real skew-symmetric $n \times n$-matrix. We carry out a discussion similar to section 4.1 for $C^\infty(M)$ to arrive at the deformed algebra $C^\infty(M)_\theta$ which will be denoted $C^\infty(M_\theta)$ (cf. [16], [17]) with completion $C(M_\theta)$. We briefly recall the following for convenience. The r-th spectral subspace for the action σ restricted to $C^\infty(M)$ is

$$C^\infty(M)_r = \{f \in C^\infty(M) : \sigma_s(f) = e^{2\pi i r \cdot s} f, \text{ for every } s \in \mathbb{T}^n\}. \tag{5.1.2}$$

Each $f \in C^\infty(M)$ can be decomposed (cf. [52]) into a unique rapidly (norm-) convergent series $f = \sum_{r \in \mathbb{Z}^n} f_r$ where $f_r \in C^\infty(M)_r$. The reference to rapid convergence implies the consideration of the family of seminorms

$$||f||_k = \sum_{r \in \mathbb{Z}^n} ||f_r||(1 + |r|)^k.$$

We give a new product $\times_\theta$ on $C^\infty(M)$ by defining it on elements of the spectral subspaces, namely for $f_r \in C^\infty(M)_r$ and $g_p \in C^\infty(M)_p$, define

$$f_r \times_\theta g_p = e^{\pi i r \cdot \theta p} f_r g_p.$$

This is then extended linearly to $C^\infty(M)$ by

$$f \times_\theta g = \sum_{r,p} e^{\pi i r \cdot \theta p} f_r g_p$$

for $f = \sum_r f_r$ and $g = \sum_p g_p$.

We now define the relevant $\mathbb{T}^n$-equivariant Θ-summable Fredholm module for $C^\infty(M_\theta)$ (cf. Definition 1.3.18) based on the spin bundle and the Dirac operator. In the following geometrical arguments we assume m to be even in order to simplify the presentation. Recall that one denotes by $SO(M)$ the principal $SO(m)$-bundle of orthonormal frames, and by $Spin(M)$ the principal $Spin(m)$-bundle which is a double cover we will denote $\psi : Spin(M) \longrightarrow SO(M)$. The action $\mathbb{T}^n \times M \longrightarrow M$ lifts to the frame bundle $\mathbb{T}^n \times SO(M) \longrightarrow SO(M)$ (through multiplication by the Jacobian, i.e. the tangent map), but in order to lift the action to $Spin(M)$ one has to go via a double covering $c : \widetilde{\mathbb{T}^n} \longrightarrow \mathbb{T}^n$. Namely, if we consider $h \in \mathbb{T}^n$ as the isometry $h : SO(M) \longrightarrow SO(M)$ it lifts to a pair of isometries $Spin(M) \longrightarrow Spin(M)$. This altogether forms the double covering $\widetilde{\mathbb{T}^n}$, which is just another copy of the n-torus. If we let $g \in \widetilde{\mathbb{T}^n}$ with $c(g) = h \in \mathbb{T}^n$, we have the diagram

$$\begin{tikzcd}
Spin(M) \arrow[r, "g"] \arrow[d, "\psi"'] & Spin(M) \arrow[d, "\psi"] \\
SO(M) \arrow[r, "h"] \arrow[d] & SO(M) \arrow[d] \\
M \arrow[r, "h"'] & M
\end{tikzcd}$$

Let $\mathbf{S}_\pm$ denote the Clifford modules on which $Spin(m)$ acts irreducibly. Recall that the spinor bundle $S = S_+ \oplus S_-$ is built from $S_\pm = Spin(M) \times_{Spin(m)} \mathbf{S}_\pm$. The induced action of $\widetilde{\mathbb{T}^n}$ on the spinor bundle S can be expressed as follows. For a section $\phi \in \Gamma(M, S)$ expressed locally over an open set $U \subset M$ by $\phi = [(\sigma, f)]$ where $\sigma : U \longrightarrow Spin(M)$ is a local spin frame field, $f : U \longrightarrow \mathbf{S}_\pm$ is a spinor-valued function, the element $g \in \widetilde{\mathbb{T}^n}$ acts on ϕ by

$$(g\phi)(x) = [(g^{-1}\sigma(x), f(h^{-1}x))], \quad x \in M.$$

For more details see e.g. [35]. We let $\mathcal{H} = L^2(M, S)$ be the Hilbert space of L^2-sections of the spinor bundle. The action by $\widetilde{\mathbb{T}^n}$ extends to give a unitary representation $U : \widetilde{\mathbb{T}^n} \longrightarrow B(\mathcal{H})$ which commutes with the Dirac operator

$$DU_s = U_s D, \quad \text{for all } s \in \widetilde{\mathbb{T}^n}. \tag{5.1.3}$$

Letting $\pi : C(M) \longrightarrow B(\mathcal{H})$ be the representation by pointwise multiplication of functions, we have

$$U_g \pi(f) U_g^* = \pi(\lambda_{c(g)}(f))$$

where $\lambda_{c(g)}(f)(x) = f(c(g)^{-1}x)$. This suggests that we consider the action $\sigma : \widetilde{\mathbb{T}^n} \longrightarrow Aut(C(M))$, $\sigma_g(f)(x) = f(c(g)^{-1}x)$, as this gives us a C*-dynamical system $(C(M), \widetilde{\mathbb{T}^n}, \sigma)$ represented on $\mathcal{H} = L^2(M, S)$ since

$$U_s \pi(f) U_s^* = \pi(\sigma_s(f)), \quad s \in \widetilde{\mathbb{T}^n}. \tag{5.1.4}$$

This puts us in a position to define a $\widetilde{\mathbb{T}^n}$-equivariant spectral triple $(C^\infty(M), \mathcal{H}, D)$ which we shall relate to the deformed algebra $C^\infty(M_\theta)$ as we now explain. First of all we use the action $\sigma : \widetilde{\mathbb{T}^n} \longrightarrow Aut(C(M))$ rather than the action (5.1.1) to construct the deformed algebra $C^\infty(M_\theta)$. As before, given homogeneous elements $f_{r_1} \in C^\infty(M)_{r_1}$ and $g_{r_2} \in C^\infty(M)_{r_2}$ we have

$$f_{r_1} \times_\theta g_{r_2} = e^{\pi i\, r_1 \cdot \theta r_2} f_{r_1} g_{r_2},$$

and this product is extended linearly to elements $f = \sum_{r_1} f_{r_1}$ and $g = \sum_{r_2} g_{r_2}$ by

$$f \times_\theta g = \sum_{r_1, r_2} f_{r_1} \times_\theta g_{r_2} = \sum_{r_1, r_2} e^{\pi i\, r_1 \cdot \theta r_2} f_{r_1} g_{r_2}.$$

Recall the representation $\varphi : C^\infty(M_\theta) \longrightarrow B(\mathcal{H})$ from Proposition 4.1.1,

$$\varphi_\theta(f) = \sum_r f_r U_{q(-\frac{\theta}{2} r)}.$$

We record some useful identities that are simple consequences of (5.1.3) and (5.1.4).

Proposition 5.1.1.

(i) $[D, fU_s] = U_s[D, \sigma_{s^{-1}}(f)]$,

(ii) $[D, fU_s]U_t = U_{st}[D, \sigma_{(st)^{-1}}(f)]$,

(iii) $[D, fU_s][D, hU_t] = U_{st}[D, \sigma_{(st)^{-1}}(f)][D, \sigma_{t^{-1}}(h)]$, *and more generally*

$$\begin{aligned} &[D, f_1 U_{s_1}][D, f_2 U_{s_2}] \cdots [D, f_k U_{s_k}] \\ &= U_{s_1 s_2 \cdots s_k}[D, \sigma_{(s_1 s_2 \cdots s_k)^{-1}}(f_1)][D, \sigma_{(s_2 \cdots s_k)^{-1}}(f_2)] \cdots [D, \sigma_{s_k^{-1}}(f_k)] \end{aligned}$$

Proof. (i)

$$\begin{aligned} [D, fU_s] &= DfU_s - fU_s D = DU_s \sigma_{s^{-1}}(f) - U_s \sigma_{s^{-1}}(f) D \\ &= U_s D \sigma_{s^{-1}}(f) - U_s \sigma_{s^{-1}}(f) D = U_s(D\sigma_{s^{-1}}(f) - \sigma_{s^{-1}}(f) D) \\ &= U_s[D, \sigma_{s^{-1}}(f)]. \end{aligned}$$

(ii)

$$\begin{aligned} [D, fU_s]U_t &= (DfU_s - fU_s D)U_t = DfU_s U_t - fU_s D U_t = DfU_{st} - fU_{st} D \\ &= [D, fU_{st}] = U_{st}[D, \sigma_{(st)^{-1}}(f)]. \end{aligned}$$

(iii)

$$[D, fU_s][D, hU_t] = [D, fU_s]U_t[D, \sigma_{t^{-1}}(h)] = U_{st}[D, \sigma_{(st)^{-1}}(f)][D, \sigma_{t^{-1}}(h)].$$

□

Using the JLO-cocycle representative (1.3.6) of the Chern character of the Θ-summable Fredholm module corresponding to the spectral triple

$$(C^\infty(M_\theta), L^2(M, S), D)$$

we have

$$\begin{aligned}
&Ch_{2k}(\varphi_\theta(f_0), \varphi_\theta(f_1), \ldots, \varphi_\theta(f_{2k})) \\
&= \int_\Delta Trace(\gamma\varphi_\theta(f_0)e^{-t_1\mathcal{D}^2}[\mathcal{D}, \varphi_\theta(f_1)]e^{-(t_2-t_1)\mathcal{D}^2} \cdots [\mathcal{D}, \varphi_\theta(f_{2k})]e^{-(1-t_{2k})\mathcal{D}^2}) \\
&dt_1 \cdots dt_{2k},
\end{aligned} \tag{5.1.5}$$

We shall relate this Chern character cocycle with the equivariant Chern character (1.3.19) cocycle by expanding the operators into their Fourier series (with respect to spectral subspaces of the action) and collecting terms using the identities of the previous proposition. Starting with smooth functions $f_0, \ldots, f_{2k} \in C^\infty(M)$, express each f_j, $j = 0, \ldots, 2k$, as $f_j = \sum_{r_j \in \mathbb{Z}^n} f_{j,r_j}$ with $f_{j,r_j} \in C^\infty(M)_{r_j}$ being the respective homogenous component. We calculate

$$\begin{aligned}
&\varphi_\theta(f_0)e^{-t_1\mathcal{D}^2}[\mathcal{D}, \varphi_\theta(f_1)]e^{-(t_2-t_1)\mathcal{D}^2} \cdots [\mathcal{D}, \varphi_\theta(f_{2k})]e^{-(1-t_{2k})\mathcal{D}^2} \\
&= \left(\sum_{r_0 \in \mathbb{Z}^n} f_{0,r_0}U_{q(-\frac{\theta}{2}r_0)}\right) e^{-t_1\mathcal{D}^2}[\mathcal{D}, \sum_{r_1 \in \mathbb{Z}^n} f_{1,r_1}U_{q(-\frac{\theta}{2}r_1)}]e^{-(t_2-t_1)\mathcal{D}^2} \cdots e^{-(t_{2k}-t_{2k-1})\mathcal{D}^2} \\
&\cdot [\mathcal{D}, \sum_{r_{2k} \in \mathbb{Z}^n} f_{2k,r_{2k}}U_{q(-\frac{\theta}{2}r_{2k})}]e^{-(1-t_{2k})\mathcal{D}^2} \\
&= \sum_{r_0,\ldots,r_{2k} \in \mathbb{Z}^n} f_{0,r_0}U_{q(-\frac{\theta}{2}r_0)}e^{-t_1\mathcal{D}^2}[\mathcal{D}, f_{1,r_1}U_{q(-\frac{\theta}{2}r_1)}]e^{-(t_2-t_1)\mathcal{D}^2} \cdots [\mathcal{D}, f_{2k,r_{2k}}U_{q(-\frac{\theta}{2}r_{2k})}] \cdots \\
&\cdots e^{-(1-t_{2k})\mathcal{D}^2} \\
&= \sum_{r_0,\ldots,r_{2k} \in \mathbb{Z}^n} f_{0,r_0}U_{q(-\frac{\theta}{2}r_0)q(-\frac{\theta}{2}r_1)\cdots q(-\frac{\theta}{2}r_{2k})}e^{-t_1\mathcal{D}^2}[\mathcal{D}, \sigma_{q(-\frac{\theta}{2}r_1)\cdots q(-\frac{\theta}{2}r_{2k})}(f_{1,r_1})]e^{-(t_2-t_1)\mathcal{D}^2} \\
&\cdots\ e^{-(t_{2k}-t_{2k-1})\mathcal{D}^2} \cdot [\mathcal{D}, \sigma_{q(-\frac{\theta}{2}r_{2k})}(f_{2k,r_{2k}})]e^{-(1-t_{2k})\mathcal{D}^2},
\end{aligned}$$

where we have used Proposition 5.1.1 and the fact that the unitaries U also commute with $e^{-t\mathcal{D}^2}$ since they commute with $\mathcal{D}$. We will collect the constants resulting from applying the group action to the homogeneous components f_{j,r_j}, $j = 0, \ldots, 2k$, as follows. Define the symbols $\lambda_m^{m'} = e^{\pi i\, m\cdot\theta m'}$, where $m, m' \in \mathbb{Z}^n$. Using this notation we get

$$\begin{aligned}
&\sigma_{q(-\frac{\theta}{2}r_1)\cdots q(-\frac{\theta}{2}r_{2k})}(f_{1,r_1}) = \lambda_{r_1}^{r_1+\ldots+r_{2k}} f_{1,r_1}, \quad \sigma_{q(-\frac{\theta}{2}r_2)\cdots q(-\frac{\theta}{2}r_{2k})}(f_{2,r_2}) = \lambda_{r_2}^{r_2+\ldots+r_{2k}} f_{2,r_2}, \\
&\ldots, \quad \sigma_{q(-\frac{\theta}{2}r_{2k})}(f_{2k,r_{2k}}) = \lambda_{r_{2k}}^{r_{2k}} f_{2k,r_{2k}}.
\end{aligned}$$

To simplify notation we define $C_{r_1,\ldots,r_{2k}} = \lambda_{r_1}^{r_1+\ldots+r_{2k}} \cdot \lambda_{r_2}^{r_2+\ldots+r_{2k}} \cdots \lambda_{r_{2k}}^{r_{2k}}$. Recall that $[\mathcal{D}, f] = df$, i.e. Clifford multiplication by the differential 1-form. It is also a fact that the unitaries U commute with this Clifford multiplication, hence we get

$$\begin{aligned}
&f_{0,r_0}U_{q(-\frac{\theta}{2}r_0)q(-\frac{\theta}{2}r_1)\cdots q(-\frac{\theta}{2}r_{2k})}e^{-t_1\mathcal{D}^2}[\mathcal{D}, \sigma_{q(-\frac{\theta}{2}r_1)\cdots q(-\frac{\theta}{2}r_{2k})}(f_{1,r_1})]e^{-(t_2-t_1)\mathcal{D}^2} \cdots \\
&\cdot e^{-(t_{2k}-t_{2k-1})\mathcal{D}^2}[\mathcal{D}, \sigma_{q(-\frac{\theta}{2}r_{2k})}(f_{2k,r_{2k}})]e^{-(1-t_{2k})\mathcal{D}^2} \\
&= C_{r_1,\ldots,r_{2k}} f_{0,r_0}U_{q(-\frac{\theta}{2}r_0)q(-\frac{\theta}{2}r_1)\cdots q(-\frac{\theta}{2}r_{2k})}e^{-t_1\mathcal{D}^2}df_{1,r_1}e^{-(t_2-t_1)\mathcal{D}^2} \cdots df_{2k,r_{2k}}e^{-(1-t_{2k})\mathcal{D}^2} \\
&= C_{r_1,\ldots,r_{2k}} f_{0,r_0}e^{-t_1\mathcal{D}^2}df_{1,r_1}e^{-(t_2-t_1)\mathcal{D}^2} \cdots df_{2k,r_{2k}}e^{-(1-t_{2k})\mathcal{D}^2}U_{q(-\frac{\theta}{2}r_0)q(-\frac{\theta}{2}r_1)\cdots q(-\frac{\theta}{2}r_{2k})}.
\end{aligned}$$

We summarize this discussion in the following lemma.

Lemma 5.1.2.

$$\begin{aligned}
&\varphi_\theta(f_0)e^{-t_1\mathcal{D}^2}[\mathcal{D},\varphi_\theta(f_1)]e^{-(t_2-t_1)\mathcal{D}^2}\cdots[\mathcal{D},\varphi_\theta(f_{2k})]e^{-(1-t_{2k})\mathcal{D}^2}\\
=&\sum_{r_0,\dots,r_{2k}\in\mathbb{Z}^n} C_{r_1,\dots,r_{2k}} f_{0,r_0}e^{-t_1\mathcal{D}^2}df_{1,r_1}e^{-(t_2-t_1)\mathcal{D}^2}\cdots df_{2k,r_{2k}}e^{-(1-t_{2k})\mathcal{D}^2}U_{q(-\frac{\theta}{2}(r_0+\dots+r_{2k}))}.
\end{aligned}$$

The following result expresses one relation between the Chern character cocycle of the deformed algebra and the equivariant Chern character cocycle, when considering the algebraic part of $C^\infty(M_\theta)$.

Theorem 5.1.3. *Let $\varphi_\theta(f_0),\dots,\varphi_\theta(f_{2k})\in C^\infty(M_\theta)$ have finite spectral subspace expansions $f_0=\sum_{r_0}^{p_0}f_{0,r_0}$, …, $f_{2k}=\sum_{r_{2k}}^{p_{2k}}f_{2k,r_{2k}}$. Then*

$$Ch_k(\varphi_\theta(f_0),\dots,\varphi_\theta(f_k))=\sum_{r_0,\dots,r_k}C_{r_1,\dots,r_k}Ch_k^{\mathbb{T}^n}(f_{0,r_0},\dots,f_{k,r_k})(q(-\frac{\theta}{2}(r_0+\dots+r_{2k}))).$$

Proof. We let $dt=dt_1\cdots dt_k$ and $C_r=C_{r_0,\dots,r_k}$. The finite sums justify the interchanging of trace, summation and integration, hence

$$\begin{aligned}
&Ch_k(\varphi_\theta(f_0),\dots,\varphi_\theta(f_k))\\
=&\int_{\Delta_k}Tr_s\left(\varphi_\theta(f_0)e^{-t_1D^2}\varphi_\theta(f_1)e^{-(t_2-t_1)D^2}\cdots\varphi_\theta(f_k)e^{-(1-t_k)D^2}\right)dt\\
=&\int_{\Delta_k}Tr_s\left(\sum_{r_0,\dots,r_k}C_rf_{0,r_0}e^{-t_1D^2}f_{1,r_1}e^{-(t_2-t_1)D^2}\cdots f_{k,r_k}e^{-(1-t_k)D^2}U_{q(-\frac{\theta}{2}(r_0+\dots+r_{2k}))}\right)dt\\
=&\int_{\Delta_k}\sum_{r_0,\dots,r_k}C_rTr_s\left(f_{0,r_0}e^{-t_1D^2}f_{1,r_1}e^{-(t_2-t_1)D^2}\cdots f_{k,r_k}e^{-(1-t_k)D^2}U_{q(-\frac{\theta}{2}(r_0+\dots+r_{2k}))}\right)dt\\
=&\sum_{r_0,\dots,r_k}C_r\int_{\Delta_k}Tr_s\left(f_{0,r_0}e^{-t_1D^2}f_{1,r_1}e^{-(t_2-t_1)D^2}\cdots f_{k,r_k}e^{-(1-t_k)D^2}U_{q(-\frac{\theta}{2}(r_0+\dots+r_{2k}))}\right)dt\\
=&\sum_{r_0,\dots,r_k}C_r\cdot Ch_k^{\mathbb{T}^n}(f_{0,r_0},\dots,f_{k,r_k})(q(-\frac{\theta}{2}(r_0+\dots+r_{2k}))).
\end{aligned}$$

□

We now recall the local formula for the equivariant Chern character cocycle. For more details on the A-hat genus and Pfaffian class which occurs in the formulas, refer to the discussion in section 5.2 and in particular the equations (5.2.2) and (5.2.5). As expressed in Theorem 1.3.20 of section 1.3, we have for the spectral triple $(C^\infty(M),L^2(S),D)$ and isometry $T:M\longrightarrow M$ that

$$\begin{aligned}
&\lim_{t\to0^+}Ch_k^G(\sqrt{t}D)(f_0,\dots,f_k)(T)\\
=&\frac{1}{k!(2\pi i)^{\frac{k}{2}}}\int_{M^T}f_0\,df_1\wedge\cdots\wedge df_k\wedge\widehat{A}(TM^T)\wedge\left(Pf(2\sin(\frac{\Omega}{4\pi}+i\frac{\Theta}{2}))(\nu(M^T))\right)^{-1}.
\end{aligned}\tag{5.1.6}$$

The following is our local formula for the Chern character of the deformed spectral triple when considering the algebraic part of $C^\infty(M_\theta)$.

Theorem 5.1.4. *For the spectral triple $(C^\infty(M_\theta), L^2(S), D)$ the Chern character is expressible as*

$$
\begin{aligned}
&Ch_k(\varphi_\theta(f_0), \ldots, \varphi_\theta(f_k)) \\
&= \sum_{r_0,\ldots,r_k} C_{r_1,\ldots,r_k} \lim_{t\to 0^+} Ch_k^{\mathbb{T}^n}(\sqrt{t}D)(f_{0,r_0}, \ldots, f_{k,r_k})(q(-\frac{\theta}{2}(r_0 + \ldots + r_{2k}))) \\
&= \frac{1}{k!(2\pi i)^{\frac{k}{2}}} \sum_{r_0,\ldots,r_k} C_{r_1,\ldots,r_k} \int_{M^{g_{r_0,\ldots,r_k}}} f_{0,r_0}\, df_{1,r_1} \wedge \cdots \wedge df_{k,r_k} \wedge \omega_{r_0,\ldots,r_k}
\end{aligned}
$$

where $f_0 = \sum_{r_0}^{p_0} f_{0,r_0}$, ..., $f_{2k} = \sum_{r_{2k}}^{p_{2k}} f_{2k,r_{2k}}$ are elements with finite spectral subspace expansions,

$$g_{r_0,\ldots,r_k} = q(-\frac{\theta}{2}(r_0 + \ldots + r_{2k})) \in \mathbb{T}^n,$$

$$C_{r_1,\ldots,r_k} = \lambda_{r_1}^{r_1+\ldots+r_{2k}} \cdot \lambda_{r_2}^{r_2+\ldots+r_{2k}} \cdots \lambda_{r_{2k}}^{r_{2k}} \text{ where } \lambda_m^{m'} = e^{\pi i m\cdot\theta m'},$$

$$\text{and } \omega_{r_0,\ldots,r_k} = \widehat{A}(TM^{g_{r_0,\ldots,r_k}}) \wedge \left(Pf(2\sin(\frac{\Omega}{4\pi} + i\frac{\Theta}{2}))(\nu(M^{g_{r_0,\ldots,r_k}}))\right)^{-1}.$$

Proof. Follows by combining Theorem 5.1.3 and (5.1.6). □

We have stated the main decomposition result above for the algebraic part of $C^\infty(M_\theta)$, i.e. smooth functions with *finite* spectral subspace expansions. In order to state such a result for the whole algebra $C^\infty(M_\theta)$, i.e. elements consisting of infinite series expansions $f = \sum_r f_r$, there are convergence issues that need to be addressed. We believe this can be achieved by a careful analysis of the needed results from [10] with a special emphasis on rate of convergence, combined with an appropriate criteria. Proposition 5.1.5 below is one step in that direction. However we shall not pursue that goal any further here, but rather in section 5.2 below we study an additional geometric condition on the isometric action, the Diophantine condition (cf. [44], [45], [46]) which does provide the means to conclude convergence of the series.

Proposition 5.1.5. *Let $f_0, \ldots, f_{2k} \in C^\infty(M)$ be smooth functions with finite spectral-subspace expansions as $f_0 = \sum_{r_0\in\mathbb{Z}^n} f_{0,r_0}, \ldots, f_{2k} = \sum_{r_{2k}\in\mathbb{Z}^n} f_{2k,r_{2k}}$. Then*

$$
\begin{aligned}
&Tr_s\left(\sum_{r_0,\ldots,r_k} f_{0,r_0} e^{-t_1 D^2} df_{1,r_1} e^{-(t_2-t_1)D^2} \cdots df_{2k,r_{2k}} e^{-(1-t_{2k})D^2} U_{q(-\frac{\theta}{2}(r_0+\ldots+r_{2k}))}\right) \\
&= \sum_{r_0,\ldots,r_k} Tr_s\left(f_{0,r_0} e^{-t_1 D^2} df_{1,r_1} e^{-(t_2-t_1)D^2} \cdots df_{2k,r_{2k}} e^{-(1-t_{2k})D^2} U_{q(-\frac{\theta}{2}(r_0+\ldots+r_{2k}))}\right)
\end{aligned}
$$

converges absolutely and uniformly in $(t_1, \ldots, t_{2k})$.

Proof. Recall the trace-class norm $||A||_1 = \sum_n \langle |A|\xi_n, \xi_n\rangle$ and more generally the p-norm $||A||_p = (\sum_n \langle |A|^p \xi_n, \xi_n\rangle)^{\frac{1}{p}}$, where $\{\xi_n\}_n \subset \mathcal{H}$ is an orthonormal basis. We will make use of the inequalities

$$|Tr(A)| \leqslant ||A||_1, \quad A \in \mathcal{L}^1(\mathcal{H}), \tag{5.1.7}$$

$$||BA||_p \leqslant ||B|| \cdot ||A||_p, \quad A \in \mathcal{L}^p(\mathcal{H}), B \in B(\mathcal{H}), \tag{5.1.8}$$

and

$$||A_1\cdots A_{2k}||_1 \leqslant ||A_1||_{p_1}\cdots||A_{2k}||_{p_{2k}}, \quad \frac{1}{p_1}+\ldots+\frac{1}{p_{2k}}=1, \quad A_j\in\mathcal{L}^{p_j}(\mathcal{H}). \quad (5.1.9)$$

We compute

$$\begin{aligned}
&\left|Tr_s\left(f_{0,r_0}e^{-t_1D^2}df_{1,r_1}e^{-(t_2-t_1)D^2}\cdots df_{2k,r_{2k}}e^{-(1-t_{2k})D^2}U_{q(-\frac{\theta}{2}(r_0+\ldots+r_{2k}))}\right)\right| \\
&\leqslant \left|\left|f_{0,r_0}e^{-t_1D^2}df_{1,r_1}e^{-(t_2-t_1)D^2}\cdots df_{2k,r_{2k}}e^{-(1-t_{2k})D^2}U_{q(-\frac{\theta}{2}(r_0+\ldots+r_{2k}))}\right|\right|_1 \\
&\leqslant \left|\left|f_{0,r_0}e^{-t_1D^2}df_{1,r_1}e^{-(t_2-t_1)D^2}\cdots df_{2k,r_{2k}}e^{-(1-t_{2k})D^2}\right|\right|_1\cdot\left|\left|U_{q(-\frac{\theta}{2}(r_0+\ldots+r_{2k}))}\right|\right| \\
&= \left|\left|f_{0,r_0}e^{-t_1D^2}df_{1,r_1}e^{-(t_2-t_1)D^2}\cdots df_{2k,r_{2k}}e^{-(1-t_{2k})D^2}\right|\right|_1
\end{aligned}$$

by (5.1.7) and (5.1.8) with $p=1$.

We put $A_1=f_{0,r_0}e^{-t_1D^2}$, $A_2=df_{1,r_1}e^{-(t_2-t_1)D^2},\ldots,A_{2k}=df_{2k,r_{2k}}e^{-(1-t_{2k})D^2}$, and $p_1=\frac{1}{t_1}$, $p_2=\frac{1}{t_2-t_1},\ldots,p_k=\frac{1}{1-t_{2k}}$, and continue to compute

$$\begin{aligned}
&\left|\left|f_{0,r_0}e^{-t_1D^2}df_{1,r_1}e^{-(t_2-t_1)D^2}\cdots df_{k,r_k}e^{-(1-t_k)D^2}\right|\right|_1 \\
&\leqslant \left|\left|f_{0,r_0}e^{-t_1D^2}\right|\right|_{p_1}\cdot\left|\left|df_{1,r_1}e^{-(t_2-t_1)D^2}\right|\right|_{p_2}\cdots\left|\left|df_{k,r_k}e^{-(1-t_k)D^2}\right|\right|_{p_k} \\
&\leqslant ||f_{0,r_0}||\cdot\left|\left|e^{-t_1D^2}\right|\right|_{p_1}\cdot||df_{1,r_1}||\cdot\left|\left|e^{-(t_2-t_1)D^2}\right|\right|_{p_2}\cdots||df_{k,r_k}||\cdot\left|\left|e^{-(1-t_{2k})D^2}\right|\right|_{p_k}
\end{aligned}$$

by first applying (5.1.9) and then applying (5.1.8) to each resulting factor. Furthermore, let $\{\xi_n\}_n\subset\mathcal{H}$ be the orthonormal eigenbasis for D with eigenvalues $\{\lambda_n\}_n$, then

$$||e^{-(t_j-t_{j-1})D^2}||_{p_j}=\left(\sum_n\left\langle\left|e^{-(t_j-t_{j-1})D^2}\right|^{p_j}\xi_n,\xi_n\right\rangle\right)^{\frac{1}{p_j}}=\left(\sum_n e^{-\lambda_n^2}\right)^{t_j-t_{j-1}} \quad (5.1.10)$$

thus

$$\begin{aligned}
&\left|\left|e^{-t_1D^2}\right|\right|_{p_1}\cdot\left|\left|e^{-(t_2-t_1)D^2}\right|\right|_{p_2}\cdots\left|\left|e^{-(1-t_{2k})D^2}\right|\right|_{p_k} \\
&=\left(\sum_n e^{-\lambda_n^2}\right)^{t_1}\cdot\left(\sum_n e^{-\lambda_n^2}\right)^{t_2-t_1}\cdots\left(\sum_n e^{-\lambda_n^2}\right)^{1-t_{2k}}=\left(\sum_n e^{-\lambda_n^2}\right)=Tr(e^{-D^2}).
\end{aligned}$$

We conclude

$$\begin{aligned}
&\left|Tr_s\left(f_{0,r_0}e^{-t_1D^2}df_{1,r_1}e^{-(t_2-t_1)D^2}\cdots df_{k,r_k}e^{-(1-t_k)D^2}U_{q(-\frac{\theta}{2}(r_0+\ldots+r_{2k}))}\right)\right| \\
&\leqslant Tr(e^{-D^2})\cdot||f_{0,r_0}||\cdot||df_{1,r_1}||\cdots||df_{k,r_k}||\,.
\end{aligned} \quad (5.1.11)$$

Denote the latter quantity

$$M_{r_0\ldots r_{2k}}=Tr(e^{-D^2})\cdot||f_{0,r_0}||\cdot||df_{1,r_1}||\cdots||df_{2k,r_{2k}}||$$

then the series

$$\sum_{r_0,\ldots,r_{2k}}M_{r_0\ldots r_{2k}}$$

converges due to the rapid decay assumption on the functions $f_0, \ldots, f_{2k}$. We now have

$$\left| Tr_s \left(f_{0,r_0} e^{-t_1 D^2} df_{1,r_1} e^{-(t_2-t_1)D^2} \cdots df_{2k,r_{2k}} e^{-(1-t_{2k})D^2} U_{q(-\frac{\theta}{2}(r_0+\ldots+r_{2k}))} \right) \right| \leqslant M_{r_0 \ldots r_{2k}},$$

so by the Weierstrass M-test we conclude that

$$\sum_{r_0,\ldots,r_k} Tr_s \left(f_{0,r_0} e^{-t_1 D^2} df_{1,r_1} e^{-(t_2-t_1)D^2} \cdots df_{2k,r_{2k}} e^{-(1-t_{2k})D^2} U_{q(-\frac{\theta}{2}(r_0+\ldots+r_{2k}))} \right)$$

converges uniformly in $(t_1, \ldots, t_{2k})$. □

5.2 Estimates on fix-point submanifolds

In this section we do some more precise geometric analysis on the terms which occur in our local formula of Theorem 5.1.4. The main interest here is the question of summability. In the previous section we restricted our attention to elements having finite spectral subspace decompositions in order to avoid convergence issues. Here we will employ an additional standing assumption, the Diophantine condition (5.2.7), which will imply convergence of the series also when it is infinite (we remark however that we have not proved the local index formula for infinite series - refer to the paragraph prior to Proposition 5.1.5). The Diophantine condition was used in the results on a local index formula for elliptic operators with shifts in [44] and [45] (see also [46]). It will provide us with the needed rate of convergence in regards to our main results from the previous section.

For an element $g \in \mathbb{T}^n$ we consider the corresponding isometry $g : M \longrightarrow M$ and its fixed point set

$$M^g = \{x \in M : gx = x\},$$

which is a closed subset of M. The following is a standard result of differential geometry.

Proposition 5.2.1. *The connected components of M^g are totally geodesic submanifolds of M.*

Proof. Let $x \in M^g$. Let $U \subset T_xM$ be a neighborhood of $0 \in T_xM$ such that $exp_x : U \longrightarrow M$ is a diffeomorphism onto $V = exp_x(U)$. We may further assume V to be strongly geodesically convex (by appropriately shrinking U if necessary), which is to say that for any two points in V there exists a unique minimizing geodesic connecting the two points. Let $U_g = \{v \in T_xM : g_{*_x} v = v\}$. We claim that

$$V \cap M^g = exp_x(U \cap U_g). \tag{5.2.1}$$

Indeed, this will follow from the fact that $g \circ exp_x = exp_x \circ g_{*_x}$ for each g (this is a standard result, for any isometry). Now let $v \in U \cap U_g$. Then $gexp_x(v) = exp_x(g_{*_x} v) = exp_x(v)$, which shows that $exp_x(U \cap U_g) \subseteq V \cap M^g$. To show the reverse inclusion, let $y \in V \cap M^g$. Then there exists a (unique) $v_y \in U$ such that $exp_x(v_y) = y$ as exp_x was a diffeomorphism $U \longrightarrow V$. We will show that $v_y \in U_g$. We calculate

$$y = gy = gexp_x(v_y) = exp_x(g_{*_x} v_y) = exp_x(v_y),$$

and we conclude, by the injectivity of exp_x in U, that $g_{*_x} v_y = v_y$, i.e. that $v_y \in U_g$ as claimed. This establishes (5.2.1). Moreover $V \cap M^g$ is geodesically convex: let $a, b \in V \cap M^g$. As $a, b \in V$ there exists a unique minimizing geodesic γ with $\gamma(0) = a$ and $\gamma(t_0) = b$ and $\gamma((0, t_0)) \subset V$. As the isometry g preserves the length of γ and $g\gamma(0) = ga = a$, $g\gamma(t_0) = gb = b$, it follows that $g \circ \gamma$ is also a minimizing geodesic connecting a and b, hence by uniqueness we must have $g \circ \gamma = \gamma$, which means $\gamma((0, t_0)) \subset M^g$. The geodesic convexity of $V \cap M^g$ implies that V intersects only one connected component of M^g. We conclude now that this connected component is exactly an embedded submanifold, since (5.2.1) expresses precisely the local criterion of being an embedded submanifold: the pairs $(V \cap M^g, exp_x^{-1})$ form an atlas. Total geodesicness follows from (5.2.1) also. $\square$

Remark 5.2.2. It follows from the compactness of M that M^g must consist of a finite number of connected components.

Working locally around a point $x \in M^g \subset M$, let $\delta(x) > 0$ be such that $exp_x : B_{\delta(x)}(0) \longrightarrow M$ is a diffeomorphism onto its image $B_{\delta(x)}(x) = exp_x(B_{\delta(x)}(0))$, where $B_{\delta(x)}(0) \subset T_x M$ is the open ball of radius $\delta(x)$. Assuming the dimension of the connected component of M^g in which $B_{\delta(x)}(x)$ lies, to be k, we can arrange an orthonormal frame $\{v_1, \ldots, v_{2k}, \ldots, v_{2n}\}$ on $B_{\delta(x)}(x)$, such that $v_1, \ldots, v_{2k}$ are fixed by the differential g_*, i.e.

$$g_{*_y} v_i(y) = v_i(y), \quad i = 1, \ldots, 2k, \quad \text{for all } y \in B_{\delta(x)}(x).$$

We let $B^g_{\delta(x)}(0)$ denote the open ball in $span\{v_1, \ldots, v_{2k}\}$ of radius $\delta(x)$, and $B^g_{\delta(x)}(x) = exp_x(B^g_{\delta(x)}(0)) \subset M^g$.

For an element $\omega \in \Gamma(\oplus_{j=1}^{2n} \Lambda^j T^* M)$, expressed in $B_{\delta(x)}(x)$ by $\omega = \sum_I \omega_I dv^I$, we put $||\omega|| = \sum_I ||\omega_I||$, where the latter $||\cdot||$ denotes the sup-norm of smooth functions. For notational brevity, put $\hat{A}(g) = \hat{A}(TM^g)$ and $Pf(g)^{-1} = Pf(2\sinh(\frac{\Omega^\perp}{4\pi} + i\frac{\Theta}{2}))^{-1}$, where $\Theta = \Theta(g)$ will be explained shortly. By definition one has

$$\hat{A}(g) = \hat{A}(TM^g) = det^{1/2}\left(\frac{\Omega^\top/4\pi}{\sinh(\Omega^\top/4\pi)}\right), \tag{5.2.2}$$

where Ω is the curvature 2-form matrix of M, expressed locally in $B_{\delta(x)}(x)$ as

$$\Omega = \begin{pmatrix} \Omega^\top & 0 \\ 0 & \Omega^\perp \end{pmatrix} \tag{5.2.3}$$

where $\Omega^\top$ ("tangential") consists of those terms which correspond to the g_*-invariant frame elements $v_1, \ldots, v_{2k}$, and $\Omega^\perp$ ("normal") of the terms corresponding to the remaining frame elements $v_{2k+1}, \ldots, v_{2n}$. The entries of the matrix Ω depend only on the curvature operator of M, and not on the specific element g; it is only the decomposition (5.2.3) which depends on the element g, i.e. it depends on $2k$ (the dimension of the connected component of M^g in which we are working locally). Hence we conclude that the function coefficients of $\hat{A}(g)$ are smooth functions on M which do not depend on g, and hence admit global bounds independently of g. This establishes

Lemma 5.2.3. *There is a constant $C_1 > 0$ such that $||\hat{A}(g)|| \leqslant C_1$.*

A more careful treatment is needed in obtaining bounds for $Pf(g)^{-1}$, as $\Theta = \Theta(g)$, the rotation matrix which represents the action of g_*, certainly depends on g, and also the occurence of the inverse needs more careful handling.

If we consider our chosen local orthonormal frame as a local section $y \mapsto V(y) = (v_1(y), \ldots, v_{2n}(y))$, for $y \in B_{\delta(x)}(x)$, then $V(y)$ can be understood as an element of $SO(2n)$. The group action lifted to the frame bundle, still denoted g_*, can then be expressed by

$$g_{*_y} V(y) = V(gy)\mathcal{T}(y) \tag{5.2.4}$$

i.e. right matrix multiplication by a matrix valued function $\mathcal{T}$, which in particular at x has the form

$$T(x) = \begin{pmatrix} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& \cos(\varphi_1) & \sin(\varphi_1) &&& \\ &&& -\sin(\varphi_1) & \cos(\varphi_1) &&& \\ &&&&& \ddots && \\ &&&&&& \cos(\varphi_{n-k}) & \sin(\varphi_{n-k}) \\ &&&&&& -\sin(\varphi_{n-k}) & \cos(\varphi_{n-k}) \end{pmatrix}$$

where $\varphi_i = \varphi_i(x)$ are functions of x. It follows from [35] that

$$Pf(g)^{-1} = Pf(2\sinh(\frac{\Omega^{\perp}}{4\pi} + i\frac{\Theta}{2}))^{-1} = \left(\prod_{j=1}^{n-k} 2\sinh(\frac{v_j^*}{2} + i\frac{\varphi_j}{2})\right)^{-1} \tag{5.2.5}$$

where the v_j^* are Chern roots of the tangential curvature matrix. Hence the relevant quantities one wishes to control the size of, are

$$(2\sinh(\frac{v_j^*}{2} + \frac{i\varphi_j}{2}))^{-1} \tag{5.2.6}$$

for $j = 1, \ldots, n-k$. We begin by establishing estimates on the φ_j. It follows from (5.2.4) that the action of g_* rotates pairs of vectors $v_{2k+2j-1}, v_{2k+2j}$ from the frame by the angle φ_j, $j = 1, 2, \ldots, n-k$. Consider now the angle φ_j, for some j. Let $z \in B_{\delta(x)}(x)$, with $z = exp_x(v_z)$ where $v_z \in span\{v_{2k+2j-1}, v_{2k+2j}\} \subset T_xM$. Then

$$gz = exp_x(R_{\varphi_j} v_z)$$

where R_{φ_j} is the rotation matrix with rotation angle φ_j. Put $v_{gz} = R_{\varphi_j} v_z$. We will employ the Diophantine assumption: there exist $C \in (0,1)$ and $N \in \mathbb{N}$ such that

$$d(q(\frac{\theta r}{2})x, x) \geqslant C|r|^{-N} dist(\{x\}, M^{q(\frac{\theta r}{2})}), \quad x \in M, r \in \mathbb{Z}^n. \tag{5.2.7}$$

We shall also use the fact that the exponential map is bi-Lipschitz with constants l and L, say

$$l||v - w||_x \leqslant d(exp_x(v), exp_x(w)) \leqslant L||v - w||_x. \tag{5.2.8}$$

The angle between the vectors v_z and v_{gz} is φ_j. We conduct the following estimate: Let s denote the arclength of the arc of the circle of radius $||v_z||$, which connects the points v_z and v_{gz}. Then by the definition of the angle between these two points,

$|\varphi_j| = \frac{s}{||v_z||}$. Clearly this arc of length s is longer than the straight line between v_z and v_{gz}, i.e. $s \geqslant ||v_{gz} - v_z||$. Assume $g = q(\frac{\theta r}{2})$ for some $r \in \mathbb{Z}^n$. Using the Diophantine assumption (5.2.7) we now get

$$\begin{aligned} |\varphi_j| = \frac{s}{||v_z||} &\geqslant \frac{||v_{gz} - v_z||}{||v_z||} \geqslant \frac{\frac{1}{L} d(z, gz)}{||v_z||} \\ &= \frac{\frac{1}{L} d(z, gz)}{d(z, x)} \geqslant \frac{\frac{1}{L} d(z, gz)}{dist(\{z\}, F)} \geqslant \frac{C}{L} |r|^{-N}. \end{aligned}$$

This gives the inequalities

$$0 < \frac{C}{L} |r|^{-N} \leqslant \varphi_j \leqslant 2\pi - \frac{C}{L} |r|^{-N} < 2\pi.$$

The right inequality is obtained by applying the same estimate to $2\pi - \varphi_j$, thus getting $2\pi - \varphi_j \geqslant \frac{C}{L}|r|^{-N}$ or equivalently $\varphi_j \leqslant 2\pi - \frac{C}{L}|r|^{-N}$.

Returning to the expression (5.2.6), one first calculates that

$$\sinh(\frac{v_j^*}{2} + \frac{i\varphi_j}{2}) = \cos(\frac{\varphi_j}{2}) \sinh(\frac{v_j^*}{2\pi}) + i \sin(\frac{\varphi_j}{2}) \cosh(\frac{v_j^*}{2}). \tag{5.2.9}$$

We then use the fact that $\cosh \geqslant 1$, that for $\zeta = a + ib$ one has $|\zeta| \geqslant |b|$ and that $\sin(x) > \frac{x(\pi - x)}{\pi}$ for $0 < x < \pi$, to get

$$\begin{aligned} \left| \sinh(\frac{v^*}{2} + \frac{i\varphi_j}{2}) \right| \geqslant |\sin(\frac{\varphi_j}{2})| &> \frac{\frac{\varphi_j}{2}(\pi - \frac{\varphi_j}{2})}{\pi} \\ &\geqslant \frac{\frac{C}{L}|g|^{-N}}{2} \frac{(\pi - (\pi - \frac{\frac{C}{L}|r|^{-N}}{2}))}{\pi} = \frac{\frac{C^2}{L^2}|r|^{-2N}}{4\pi}. \end{aligned}$$

This produces the bound

$$\left| 2\sinh(\frac{v^*}{2} + \frac{i\varphi_j}{2}) \right|^{-1} \leqslant \frac{2\pi L^2 |r|^{2N}}{C^2},$$

which in turn yields

Lemma 5.2.4.

$$||Pf(g)^{-1}|| \leqslant \left(\frac{2\pi L^2 |r|^{2N}}{C^2} \right)^{n-k}$$

where $g = q(\frac{\theta r}{2})$ for some $r \in \mathbb{Z}^n$.

Suppose we are given smooth functions $f_0, \ldots, f_{2k} \in C^\infty(M)$. Still working with the local orthonormal frame $\{v_1, \ldots, v_{2k}, \ldots, v_{2n}\}$ around $x \in M^g$, we may locally express the $2k$-degree term of $f_0 df_1 \wedge \cdots \wedge df_{2k} \wedge \hat{A}(g) \wedge Pf(g)^{-1}$ restricted to $B^g_{\delta(x)}(x)$ by

$$h\, dv = h\, dv^1 \wedge \cdots \wedge dv^{2k} \tag{5.2.10}$$

for some $h \in C^\infty(B^g_{\delta(x)}(x))$, with

$$||h\, dv|| \leqslant ||f_0|| \cdot ||df_1|| \cdots ||df_{2k}|| \cdot C_1 \cdot \left(\frac{2\pi L^2 |r|^{2N}}{C^2} \right)^{n-k}.$$

The exponential map exp_x provides coordinates $y_1, \dots, y_{2n}$ for any $y \in B_{\delta(x)}(x)$ by $y = exp_x(\sum_{j=1}^{2n} y_i v_j(x))$, where $\sum_{j=1}^{2n} y_i v_j(x) \in B_{\delta(x)}(0)$. For any $z \in B^g_{\delta(x)}(x)$ the coordinates are $(z_1, \dots, z_{2k})$ with $z = exp_x(\sum_{j=1}^{2k} z_i v_j(x))$, and $\sum_{j=1}^{2k} z_i v_j(x) \in B^g_{\delta(x)}(0)$. Hence, in the local coordinates in $B^g_{\delta(x)}(x)$, h is a function $(z_1, \dots, z_{2k}) \mapsto h(z_1, \dots, z_{2k})$. We recall that $B^g_{\delta(x)}(0) \subset T_x M$ is a $2k$-dimensional ball embedded in the $2n$-dimensional ball $B_{\delta(x)}(0) \subset T_x M$. Thus, in local coordinates, we may extend h on $B^g_{\delta(x)}(0)$ to the function $\bar{h}$ on $B_{\delta(x)}(0)$ by putting

$$\bar{h}(a_1, \dots, a_{2k}, \dots, a_{2n}) = h(a_1, \dots, a_{2k}), \qquad \text{for } \sum_{j=1}^{2n} a_j v_j(x) \in B_{\delta(x)}(0),$$

and we may thus consider the $2n$-form $\bar{h}\, dv^1 \wedge \cdots \wedge dv^{2n}$ on $B_{\delta(x)}(x)$.

We now want to compare the integrals of h and $\bar{h}$. First we observe the following: integration of the $2n$-form on M using exponential coordinates provides a measure μ_1, while integration of the $2n$-form on M using usual charts provides a measure λ_1. These measures are mutually absolutely continuous, and the Radon-Nikodym derivative denoted $b_1 = \frac{d\mu_1}{d\lambda_1}$ is a smooth function on M. One carries out the same considerations on M^g: integration of the $2k$-form on M^g using exponential coordinates provides a measure μ_2, while integration of the $2k$-form on M^g using usual charts provides a measure λ_2. But now we take the Radon-Nikodym derivative the other way: $b_2 = \frac{d\lambda_2}{d\mu_2}$. Let P be a parametrization of the domain of integration $B_{\delta(x)}(0)$, and let

$$R = \int_P dv^{2k+1} \cdots dv^{2n}. \tag{5.2.11}$$

$$\int_{B^g_{\delta(x)}(x)} |h| dv^1 \wedge \cdots \wedge dv^{2k} = \int_{B^g_{\delta(x)}(0)} |b_2 h| dv^1 \cdots dv^{2k} \tag{5.2.12}$$

$$\leqslant ||b_2|| \int_{B^g_{\delta(x)}(0)} |h| dv^1 \cdots dv^{2k} \tag{5.2.13}$$

$$= ||b_2|| \frac{1}{R} \int_{B_{\delta(x)}(0)} |\bar{h}| dv^1 \cdots dv^{2k} \cdots dv^{2n} \tag{5.2.14}$$

$$= ||b_2|| \frac{1}{R} \int_{B_{\delta(x)}(x)} |b_1 \bar{h}| dv^1 \wedge \cdots \wedge dv^{2k} \wedge \cdots \wedge dv^{2n} \tag{5.2.15}$$

$$\leqslant ||b_2|| \cdot ||b_1|| \cdot ||h|| \cdot \lambda_1(B_{\delta(x)}(x)) \leqslant ||b_2|| \cdot ||b_1|| \cdot ||h|| \cdot vol(M). \tag{5.2.16}$$

One uses $\int_{M^g} d\lambda_2 = \int_{M^g} b_2 d\mu_2$ to get (5.2.12). The passage from (5.2.13) to (5.2.14) uses the fact that $\bar{h}$ doesn't depend on the last $2n - 2k$ variables and hence the introduction of the constant R. The passage from (5.2.14) to (5.2.15) then uses $\int_M d\mu_1 = \int_M b_1 d\lambda_1$.

We now arrive at the final estimates. Let $f_0, \dots, f_{2k} \in C^\infty(M)$ be given. Then we want to estimate

$$\left| \int_{M^g} f_0 df_1 \wedge \cdots \wedge df_{2k} \wedge \hat{A}(g) \wedge Pf(g)^{-1} \right|.$$

We work with the open cover $\{B^g_{\delta(x)}(x)\}_{x\in M^g}$ of M^g, and take a subordinate partition of unity $\{\psi_x\}_{x\in M^g}$. By the compactness of M^g one extracts the finite subcover

$$B^g_{\delta_1}(x_1), \ldots, B^g_{\delta_p}(x_p)$$

with corresponding partition of unity functions $\psi_1, \ldots, \psi_p$, for certain $x_i \in M^g$, $i = 1, \ldots p$. Then

$$\int_{M^g} f_0 df_1 \wedge \cdots \wedge df_{2k} \wedge \hat{A}(g) \wedge Pf(g)^{-1} = \sum_{j=1}^{p} \int_{B^g_{\delta_j}(x_j)} \psi_j h_j dv^{1,j} \wedge \cdots \wedge dv^{2k,j}$$

where h_j is the local expression as in (5.2.10) over the neighborhood $B^g_{\delta_j}(x_j)$, for which we denote by $v_{1,j}, \ldots v_{2k,j}, \ldots, v_{2n,j}$ the local orthonormal frame and by

$$dv^{1,j}, \ldots, dv^{2k,j}, \ldots, dv^{2n,j}$$

the corresponding dual frame. Here it will be implicit that $2k = 2k(j)$ is the dimension of the connected component to which $B^g_{\delta_j}(x_j)$ belongs (i.e. k depends on j), though the j will be suppressed. As before we also extend the ψ_j on $B^g_{\delta_j}(x)$ to $\bar{\psi}_j$ on $B_{\delta_j}(x)$, and moreover we know that $\psi_j \leqslant 1$. As in (5.2.11) we get constants R_j, $j = 1, \ldots, p$. Each R_j depends on the radius δ_j and on the dimension of the connected component of M^g which contains x_j. The δ_j depends only on the local topology of M, as it was taken to ensure local diffeomorphism property of exp and geodesic convexity. And there are only finitely many possibilities for the dimensions of the connected components, i.e. $1 \leqslant dim(conn.comp.) \leqslant dim(M)$. Hence the constants R_j do not depend on the element g, and moreover we can find the smallest and largest constants among the R_j, denoting them R and S respectively. Then one calculates

$$\begin{aligned}
&\left|\int_{M^g} f_0 df_1 \wedge \cdots \wedge df_{2k} \wedge \hat{A}(g) \wedge Pf(g)^{-1}\right| = \left|\sum_{j=1}^{p} \int_{B^g_{\delta_j}(x_j)} \psi_j h_j \, dv^{1,j} \wedge \cdots \wedge dv^{2k,j}\right| \\
&\leqslant \frac{1}{R}||h|| \cdot vol\left(\cup_{j=1}^{p} B_{\delta_j}(0)\right) \leqslant \frac{1}{R}||h|| vol(M) \\
&= \frac{1}{R}||f_0|| \cdot ||df_1|| \cdots ||df_{2k}|| \cdot C_1 \cdot \left(\frac{2\pi L^2 |r|^{2N}}{C^2}\right)^{n-k} \cdot vol(M).
\end{aligned}$$

We summarize the discussion in the following theorem.

Theorem 5.2.5. *Let M be a compact Riemannian manifold with an action of $\mathbb{T}^n$. Assume the action satisfies the Diophantine property: there exists $C > 0$ and $N \in \mathbb{N}$ such that $d(q(\frac{\theta r}{2})x, x) \geqslant C|r|^{-N} dist(\{x\}, M^{q(\frac{\theta r}{2})})$, for $x \in M$, $r \in \mathbb{Z}^n$. Then there exist constants $R > 0$ and $C_1 > 0$ such that for any $g = q(\frac{\theta r}{2}) \in \mathbb{T}^n$ ($r \in \mathbb{Z}^n$)*

$$\begin{aligned}
&\left|\int_{M^g} f_0 df_1 \wedge \cdots \wedge df_{2k} \wedge \hat{A}(g) \wedge Pf(g)^{-1}\right| \\
&\leqslant \frac{1}{R}||f_0|| \cdot ||df_1|| \cdots ||df_{2k}|| \cdot C_1 \cdot \left(\frac{2\pi L^2 |r|^{2N}}{C^2}\right)^{n-k} \cdot vol(M).
\end{aligned}$$

Now that we have an estimate on the integral, we may consider a dense *-subalgebra of $C^\infty(M)$ consisting of functions which satisfy an appropriate growth condition, e.g. those $f \in C^\infty$ for which $\sum_{r \in \mathbb{Z}^n} ||f_r||(1+|r|^2)^k < \infty$. For such elements we can, under the assumption of the Diophantine condition, conclude that the series

$$\frac{1}{k!(2\pi i)^{\frac{k}{2}}} \sum_{r_0,\dots,r_k} C_{r_1,\dots,r_k} \int_{M^{g_{r_0,\dots,r_k}}} f_{0,r_0}\, df_{1,r_1} \wedge \cdots \wedge df_{k,r_k} \wedge \omega_{r_0,\dots,r_k}$$

of Theorem 5.1.4 is indeed a convergent series.

Bibliography

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer. *Deformation theory and quantization. I. Deformations of symplectic structures*, Ann. Physics 111 (1978), no. 1, 61-110.

[2] J. Bhowmick, S. Neshveyev, A. Sangha. *Deformation of operator algebras by Borel cocycles*, Journal of Functional Analysis, volume 265, issue 6 (2013), p.983-1001

[3] B. Blackadar. *K-theory for operator algebras*, 2nd edition, Mathematical Sciences Research Institute Publications, 5. Cambridge University Press, Cambridge, 1998

[4] J. Block, J. Fox. *Asymptotic Pseudodifferential operators and Index theory*, Contemporary Mathematics Vol 89 (1989)

[5] R. R. Busby, H. A. Smith. *Representations of twisted group algebras*, Trans. Amer. Math. Soc. 149 (1970), 503-537

[6] A. Carey, J. Phillips, A. Rennie. *Spectral triples: examples and index theory*, Noncommutative Geometry and Physics: Renormalisation, Motives, Index Theory, European Mathematical Society Publishing House, Zurich

[7] A. Carey, J. Phillips, A. Rennie, F. Sukochev. *The local index theorem in semifinite von Neumann algebras I*

[8] A. Carey, J. Phillips, A. Rennie, F. Sukochev. *The local index theorem in semifinite von Neumann algebras II*

[9] F. Combes. *Crossed products and Morita equivalence*, Proc. London Math. Soc. (3) 49 (1984), no. 2, p.289-306

[10] S. Chern, X. Hu. *Equivariant Chern character for the invariant Dirac operator*, Michigan Math J. 44 (1997).

[11] A. Connes. *An analogue of the Thom isomorphism for crossed products of C^*-algebras by an action of* $\mathbb{R}$, Advances in mathematics 39 (1981), p. 39-55

[12] A. Connes. *Non-commutative differential geometry, Parts I and II*, Publ. I.H.E.S., 62 (1985), p.257-360.

[13] A. Connes. *Noncommutative geometry*, Academic Press (1994)

[14] A. Connes, H. Moscovici. *Transgression and the Chern character of finite-dimensional K-cycles*, Commun. Math. Phys. 155, p.103-122 (1993).

[15] A. Connex, H. Moscovici. *The local index formula in noncommutative geometry*, GAFA 5 (1995)

[16] A. Connes, M. Dubois-Violette. *Noncommutative finite-dimensional manifolds I. Spherical manifolds and related examples*, preprint arXiv:math/0107070v5

[17] A. Connes, G. Landi. *Noncommutative manifolds, the instanton algebra and isospectral deformations*

[18] J. Dixmier. *C*-algebras*, North-Holland mathematical library v.15, 1977.

[19] S. Echterhoff, S. Kaliszewski, J. Quigg, I. Raeburn. *A Categorical Approach to Imprimitivity Theorems for C*-Dynamical Systems*, vol. 180, Mem. Amer. Math. Soc., no. 850, AMS, Providence, RI, 2006

[20] S. Echterhoff, W. Lück, N.C. Phillips, S. Walters, *The structure of crossed products of irrational rotation algebras by finite subgroups of* $\mathrm{SL}_2(\mathbb{Z})$, J. Reine Angew. Math. **639** (2010), 173–221.

[21] S. Echterhoff, R. Nest, H. Oyono-Oyono. *Principal noncommutative torus bundles*, arXiv:0810.0111v1 [math.KT], (2008)

[22] S. Echterhoff, R. Nest, H. Oyono-Oyono. *Fibrations with noncommutative fibers*, arXiv:0810.0118v1 [math.KT],(2008)

[23] G. A. Elliott, T. Natsume, R. Nest. *Cyclic cohomology for one-parameter smooth crossed products*, Acta Mathematica Volume 160, Number 1 (1988), p.285-305

[24] K.C. Hannabuss, V. Mathai, *Noncommutative principal torus bundles via parametrised strict deformation quantization*, in: "Superstrings, geometry, topology, and C^*-algebras", 133–147, Proc. Sympos. Pure Math., **81**, Amer. Math. Soc., Providence, RI, 2010.

[25] G. Hao, C. K. Ng. *Crossed products of C*-correspondences by amenable group actions*, J. Math. Anal. Appl. 345 (2008) 702-707

[26] N. Higson. *The residue index theorem of Connes and Moscovici*, Clay Mathematics Proceedings Volume 6 (2006)

[27] K. K. Jensen, K. Thomsen. *Elements of KK-theory*, Birkhuser Boston, Inc., Boston, MA, 1991

[28] S. Kaliszewski, J. Quigg, D. Robertson. *Functoriality of Cuntz-Pimsner correspondence maps*, arXiv:1204.5820v1 [math.OA], (2012)

[29] S. Kaliszewski, J. Quigg, D. Robertson. *Coactions on Cuntz-Pimsner algebras*, arXiv:1204.5822v1 [math.OA], (2012)

[30] G. G. Kasparov. *Equivariant KK-theory and the Novikov conjecture*, Invent. math 91, p147-201 (1988)

[31] P. Kasprzak, *Rieffel deformation via crossed products*, J. Funct. Anal. **257** (2009), no. 5, 1288–1332.

[32] T. Katsura. *On C*-algebras associated with C*-correspondences*, preprint, arXiv:math/0309088v3 (2003)

[33] E. Kirchberg, S. Wassermann. *Operations on continuous bundles of C*-algebras*, Math. Ann. 303(1995), p.677-697

[34] D. Kaschek, N. Neumaier and S. Waldmann, *Complete positivity of Rieffel's deformation quantization by actions of* $\mathbb{R}^d$, J. Noncommut. Geom. **3** (2009), no. 3, 361–375.

[35] J. D. Lafferty, Y. Yu, W. Zhang, *A Direct Geometric Proof of the Lefschetz Fixed Point Formulas*, Transactions of the American Mathematical Society, Vol. 329, No. 2 (Feb., 1992), pp.571-583

[36] M.B. Landstad, J. Phillips, I. Raeburn, C.E. Sutherland. *Representations of crossed products by coactions and principal bundles*, Trans. Amer. Math. Soc. **299** (1987), no. 2, 747–784.

[37] M. B. Landstad. *Duality theory for covariant systems*, Trans. Amer. Math. Soc. 248 (1979), no. 2, 223-267

[38] G. Landi. *Examples of noncommutative instantons*

[39] K. Matsumoto. *Smooth structures, actions of Lie algebra* $\int u(2)$ *and Haar measures on non-commutative three dimensional spheres*m Tokyo J. Math. Vol. 15 No. 1 (1992)

[40] C.C. Moore, *Extensions and low dimensional cohomology theory of locally compact groups. I*, Trans. Amer. Math. Soc. **113** (1964), 40–63.

[41] C.C. Moore, *Group extensions and cohomology for locally compact groups. III*, Trans. Amer. Math. Soc. **221** (1976), no. 1, 1–33.

[42] P. Muhly, B. Solel. *Tensor algebras over C*-correspondences: Representations, Dilations and C*-envelopes*, Journal of functional analysis, 158 (1998)

[43] G. Nagy. *Deformation quantization and K-theory*, AMS Contemporary Mathematics Series, vol. 214 Perspectives on quantization (1998), p111-135

[44] V. E. Nazaikinskii, A. Yu. Savin, B. Yu. Sternin. *On elliptic differential operators with shifts*,

[45] V. E. Nazaikinskii, A. Yu. Savin, B. Yu. Sternin. *On elliptic differential operators with shifts II. The cohomological index formula*,

[46] V. E. Nazaikinskii, A. Yu. Savin, B. Yu. Sternin. *Elliptic theory and noncommutative geometry. Nonlocal operators.*, Operator theory: advances and applications. Vol. 183. Birkhuser (2008)

[47] R. Nest. *Cyclic cohomology of crossed products with* $\mathbb{Z}$. Journal of Functional Analysis, 80, 235-283 (1988)

[48] C. K. Ng. *Morita equivalences between fixed point algebras and crossed products*, Math. Proc. Cambridge Philos. Soc. 125 (1999), p.43-52

[49] J.A. Packer and I. Raeburn, *Twisted crossed products of C^*-algebras*, Math. Proc. Cambridge Philos. Soc. **106** (1989), no. 2, 293–311.

[50] G. K. Pedersen. *C^*-algebras and their automorphism groups*, Academic Press (1979)

[51] J.C. Quigg, *Landstad duality for C^*-coactions*, Math. Scand. **71** (1992), no. 2, 277–294.

[52] M. A. Rieffel. *Deformation quantization for actions of* $\mathbb{R}^d$, Mem. Am. Math. Soc. 506 (1993)

[53] M. A. Rieffel. *K-groups of C^*-algebras deformed by actions of* $\mathbb{R}^d$, Journal of functional analysis 116, (1993), p.199-214.

[54] M. A. Rieffel. *Proper actions of groups on C^*-algebras*, Mappings of operator algebras, Birkhauser, (1990)

[55] A. Sangha. *KK-fibrations arising from Rieffel deformations*, preprint, arXiv:1109.5968 [math.OA]

[56] K. Thomsen. *KK-equivalence and continuous bundles of C^*-algebras*, New York J. Math. 6 (2000) p285-306.

[57] S. Vaes. *A new approach to induction and imprimitivity results*, Journal of Functional Analysis, 229, p.317-374 (2005)

[58] S. Vaes, *Strictly outer actions of groups and quantum groups*, J. Reine Angew. Math. **578** (2005), 147–184.

[59] J. C. Varilliy, H. Figueroa, J. M. Gracia-Bondia. *Elements of Noncommutative Geometry*, Birkhuser (2000)

[60] D. Williams. *Crossed products of C^*-algebras* Mathematical Surveys and Monographs Volume 134, AMS (2007)

[61] M. Yamashita. *Connes-Landi deformation of spectral triples*, preprint arXiv:1006.4420v1

[62] M. Yamashita, *Deformation of algebras associated to group cocycles*, preprint arXiv:1107.2512v1 [math.OA].

[63] A. Yashinski, *The Gauss-Manin connection and noncommutative tori*, preprint arXiv:1210.4531v1 [math.KT].